On the 'Ordnungszahlen' in Gentzen's First Consistency Proof

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Introduction

[1] Gentzen, Die Widerspruchsfreiheit der reinen Zahlentheorie. Math. Ann. 112 (1936).

[2] Gentzen, Neue Fassung des Widerspruchsfreiheitsbeweises für die reine Zahlentheorie. Forsch. zur Logik u. zur Grundl. d. ex. Wiss. Neue Folge 4 (1938)

[3] Kogan-Bernstein, Simplification of Gentzen's reductions in the classical arithmetic. Zap.Nauch.Sem. LOMI 105 (1981) English translation: J.MATH.SCI. Vol.22(3) (1983)

§1 Derivations [1]

§2 Gentzen's 'Ordnungszahlen' $(\mathcal{O}, <_{\mathbb{R}})$ [1]

§3 Assignment of 'Ordnungszahlen' to derivations [1]

§4 Embedding of $(\mathcal{O}, <_{\mathbb{R}})$ into $(On, <)$

§5 Transforming Gentzen's assignment into the assignment in [3]

§6 Reduction steps on sequents [1]

§7 Reduction steps on derivations [1]

§8 Transition to multisuccedent seqents (LK) yielding an "amalgamation" of [1] and [2]

§1 Derivations. Inductive Definition of $d \vdash \Pi$.

1. Atomic derivations (axioms)
2. $d_0 \vdash \Gamma \to F(a)$ & variable condition $\Longrightarrow \mathrm{I}^a_{\forall x F} d_0 \vdash \Gamma \to \forall x F(x)$.
3. $d_0 \vdash \Gamma, A \to \bot \Longrightarrow \mathrm{I}_{\neg A} d_0 \vdash \Gamma \to \neg A$.
4. $d_0 \vdash \Gamma \to F(0)$ & $d_1 \vdash F(a), \Delta \to F(\mathsf{S}a)$ & var.cond. $\Longrightarrow \mathrm{Ind}^{a,t}_F d_0 d_1 \vdash \Gamma, \Delta \to F(t)$
5. If $d_i \vdash \Pi_i$ $(i = 0, \ldots, l)$, and if $\dfrac{\Pi_0 \ \ \ldots \ \ \Pi_l}{\Pi}$ is a *chain inference of rank* r, then $\mathbf{K}^r_\Pi d_0 \ldots d_l \vdash \Pi$.

§2 Gentzen's 'Ordnungszahlen' $(\mathcal{O}, <_{\mathbb{R}})$

$u, v \in \{0,1\}^+$, $\quad 0^n := \overbrace{0 \ldots 0}^{n}$, $\quad n.u \in \mathbb{R}$

Definition of $M_n \subseteq \{0,1\}^+$ (Mantissen)

1. $M_0 := \{1\}$;
2. $M_{n+1} := \{u_0 0^{n+1} u_1 0^{n+1} \ldots 0^{n+1} u_l : u_0, \ldots, u_l \in M_n \ \& \ 0.u_l \leq_{\mathbb{R}} \ldots \leq_{\mathbb{R}} 0.u_0\}$.

$M := \bigcup_{n \in \mathbb{N}} M_n$, $\quad \mathsf{h}(u) := \min\{n : u \in M_n\}$

Remarks. (a) $M_n \subseteq M_{n+1}$.

(b) $\mathsf{h}(u)$ is the maximal number of consecutive zeros in u (*<u>H</u>öchstanzahl aufeinanderfolgender Nullen*).

Definition.

$\mathcal{O} := \{n.u : n \in \mathbb{N} \ \& \ u \in M_n\}$ (Ordnungszahlen)

Theorem 1. $\mathcal{O}$ is wellordered by $<_{\mathbb{R}}$.

§3 Assignment of 'Ordnungszahlen' to derivations

Definition of $\rho(d) \in \mathbb{N}$ (Numerus) and $\mu(d) \in M$ (Mantisse)

Case $d = \mathbf{K}^r_\Pi d_0 \dots d_l$ (w.l.o.g. $l > 0$).

Let σ be a permut. s.t. $0.\mu(d_{\sigma(0)}) \geq_{\mathbb{R}} \dots \geq_{\mathbb{R}} 0.\mu(d_{\sigma(l)})$.

$\mu(d) := u_0 0^{\nu+1} u_1 0^{\nu+1} \dots 0^{\nu+1} u_l$,

where $u_i := \mu(d_{\sigma(i)})$ and $\nu := \max\{\mathsf{h}(u_0), \dots, \mathsf{h}(u_l)\}$

Abbreviation. $\mathsf{h}(d) := \mathsf{h}(\mu(d))$, $\mathsf{h}'(d) := \mathsf{h}(d) - 1$.

Proposition. $\mathsf{h}'(d) = \max\{\mathsf{h}(d_0), \dots, \mathsf{h}(d_l)\}$ $(***)$

The numerus $\rho(d)$ is defined by the equation
$\rho(d) - \mathsf{h}(d) = \max(\{(\rho(d_i) - \mathsf{h}(d_i)) - 1 : i \leq l\} \cup \{r\})$.

Remark.
With $\mathsf{dg}(d) := \rho(d) - \mathsf{h}(d)$, this equation becomes
$\mathsf{dg}(d) = \max(\{\mathsf{dg}(d_i) - 1 : i \leq l\} \cup \{r\})$.

Definition.
$\mathsf{Ord}(d) := \rho(d).\mu(d)$ (*Ordnungszahl* of d)

Since (by definition) $\mathsf{h}(\mu(d)) = \mathsf{h}(d) \leq \rho(d)$,

we have $\mu(d) \in M_{\rho(d)}$, i.e. $\underline{\mathsf{Ord}(d) \in \mathcal{O}}$.

Theorem 2. If d^- results from d by a *reduction step on derivations* then $\mathsf{Ord}(d^-) <_{\mathbb{R}} \mathsf{Ord}(d)$.

§4 Embedding of $(\mathcal{O}, <_{\mathbb{R}})$ into $(On, <)$

Definition of $|u|_n \in On$ **for** $u \in M_n$

1. $|1|_0 := 0$.
2. $|u_0 0^{n+1} \dots 0^{n+1} u_l|_{n+1} := \omega^{|u_0|_n} + \dots + \omega^{|u_l|_n}$.

Lemma 1. For $u \in M_n$ the following holds:
(a) $|u|_{n+k} = \omega_k(|u|_n)$,
(b) $\omega_n(0) \leq |u|_n < \omega_{n+1}(0)$.

Definition. For $n.u \in \mathcal{O}$ let $|n.u| := |u|_n$.

Lemma 2.
$n.u,\, m.v \in \mathcal{O}$ & $n.u <_{\mathbb{R}} m.v \Rightarrow |n.u| < |m.v|$.

Definition. $\mathsf{o}(d) = |\mathsf{Ord}(d)|$.

Theorem 2 together with Lemma 2 yields: $\mathsf{o}(d^-) < \mathsf{o}(d)$.

Our goal is, to find a direct recursive definition of $\mathsf{o}(d)$ which does not use the assignment $\mathsf{Ord}(d)$.

§5 Transforming Gentzen's assignment

Abbreviation. $\widetilde{\mathsf{o}}(d) := |\mu(d)|_{\mathsf{h}(d)}$

Lemma 3. $\mathsf{o}(d) = \omega_{\mathsf{dg}(d)}(\widetilde{\mathsf{o}}(d))$.

Proof:
$\mathsf{o}(d) = |\rho(d).\mu(d)| = |\mu(d)|_{\rho(d)} \overset{\text{L.1a}}{=} \omega_{\rho(d)-\mathsf{h}(d)}(|\mu(d)|_{\mathsf{h}(d)})$.

Lemma 4.
For $d = \mathbf{K}^r_\Pi d_0 \dots d_l$ we have $\widetilde{\mathsf{o}}(d) = \omega^{\alpha_0} \# \dots \# \omega^{\alpha_l}$
with $\alpha_i := \omega_{\mathsf{h}'(d)-\mathsf{h}(d_i)}(\widetilde{\mathsf{o}}(d_i))$

Lemma 4.
For $d = \mathbf{K}^r_\Pi d_0 \dots d_l$ we have $\widetilde{\mathsf{o}}(d) = \omega^{\alpha_0} \# \dots \# \omega^{\alpha_l}$
with $\alpha_i := \omega_{\mathsf{h}'(d)-\mathsf{h}(d_i)}(\widetilde{\mathsf{o}}(d_i))$

Proof:
By definition, $\mu(d) = u_0 0^{\nu+1} \dots 0^{\nu+1} u_l$ with
$u_{\sigma'(i)} = \mu(d_i)$ and $\nu = \max\{\mathsf{h}(u_0), \dots, \mathsf{h}(u_l)\}$.

Hence $\nu = \max\{\mathsf{h}(d_0), \dots, \mathsf{h}(d_l)\} = \mathsf{h}'(d) = \mathsf{h}(d) - 1$,

$\widetilde{\mathsf{o}}(d) \overset{\text{Def}}{=} |\mu(d)|_{\mathsf{h}(d)} = |\mu(d)|_{\nu+1} = \omega^{|u_0|_\nu} + \dots + \omega^{|u_l|_\nu}$,

$|u_{\sigma'(i)}|_\nu = |\mu(d_i)|_\nu \overset{\text{L.1a}}{=} \omega_{\nu-\mathsf{h}(d_i)}(|\mu(d_i)|_{\mathsf{h}(d_i)})$.

Summary. For $d = \mathbf{K}^r_\Pi d_0 \dots d_l$ we have

- $\mathsf{dg}(d) = \max(\{\mathsf{dg}(d_i){-}1 : i \le l\} \cup \{r\}$
- $\mathsf{h}(d) = \max\{\mathsf{h}(d_0), \dots, \mathsf{h}(d_l)\} + 1$
- $\widetilde{\mathsf{o}}(d) = \omega^{\alpha_0} \# \dots \# \omega^{\alpha_l}$ with $\alpha_i := \omega_{\mathsf{h}'(d)-\mathsf{h}(d_i)}(\widetilde{\mathsf{o}}(d_i))$
- $\mathsf{o}(d) = \omega_{\mathsf{dg}(d)}(\widetilde{\mathsf{o}}(d))$

Idea. Simplify this ordinal assignment by setting

$$\widetilde{\mathsf{o}}(d) := \omega^{\widetilde{\mathsf{o}}(d_0)} \# \dots \# \omega^{\widetilde{\mathsf{o}}(d_l)}$$

Then $\mathsf{h}(d)$ *becomes obsolete, and the proof of* $\mathsf{o}(d^-) < \mathsf{o}(d)$ *works as well with this simpler* $\widetilde{\mathsf{o}}$.

Comparison with [3]: $\mathsf{h}(d) =$ excess of d, $\widetilde{\mathsf{o}}(d) = \mathrm{FO}(d)$

Preliminaries to part II (Gentzen 1936)

- Formulas A, B, C, F are build up from arithmetic prime formulas by $\forall$, $\wedge$, $\neg$.
  In the following, $\wedge$ is not mentioned.
- Sequents: $\Pi = \Gamma \to C$,

  $\mathsf{L}(\Gamma \to C) := \Gamma$, $\mathsf{R}(\Gamma \to C) := \{C\}$,

  $A, \Pi := A, \Gamma \to C = (\{A\} \cup \Gamma) \to C$,

  $\Pi, A := \Gamma \to A$ **!!!**
- Π has **endform** $:\Leftrightarrow \top \in \mathsf{R}(\Pi) \vee \bot \in \mathsf{L}(\Pi)$.

§6 Reduction steps on sequents

$$(\mathsf{R}_{\forall x F})\ \frac{\dots \Gamma \to F(n) \dots (n \in \mathbb{N})}{\Gamma \to \forall x F(x)} \quad \textbf{(‘Wahlfreiheit’)}$$

$$(\mathsf{L}^k_{\forall x F})\ \frac{F(k), \Gamma \to C}{\Gamma \to C} \ \text{ if } \forall x F \in \Gamma$$

$$(\mathsf{R}_{\neg A})\ \frac{A, \Gamma \to \bot}{\Gamma \to \neg A} \qquad (\mathsf{L}^0_{\neg A})\ \frac{\Gamma \to A}{\Gamma \to C} \ \text{ if } \neg A \in \Gamma$$

$$\mathcal{I}(\Pi, n) := \begin{cases} \Pi, F(n) & \text{if } \mathcal{I} = \mathsf{R}_{\forall x F} \\ F(k), \Pi & \text{if } \mathcal{I} = \mathsf{L}^k_{\forall x F} \\ A, \Pi, \bot & \text{if } \mathcal{I} = \mathsf{R}_{\neg A} \\ \Pi, A & \text{if } \mathcal{I} = \mathsf{L}^0_{\neg A} \end{cases}$$

Definition of $\mathcal{I} \rhd \Pi$ ($\mathcal{I}$ is permissible for Π)

$\mathsf{R}_B \rhd \Pi \ :\Leftrightarrow\ B \in \mathsf{R}(\Pi)$

$\mathsf{L}^k_B \rhd \Pi \ :\Leftrightarrow\ B \in \mathsf{L}(\Pi)$

Gentzen's Kettenschluß

$\dfrac{\Gamma_0 \to A_0 \ \dots \ \Gamma_l \to A_l}{\Gamma \to C}$ is a chain inference of degree r if there exists a $j \le l$ such that $A_j = C$ and $\forall i \le j(\Gamma_i \subseteq \Gamma, A_0, \dots, A_{i-1})$ and $\forall i < j(\mathrm{rk}(A_i) \le r)$.

§7 Reduction steps on derivations

For each derivation d, whose endsequent is not in endform we shall now define *the reduction step on* d and at the same time **prove** the following:

by such a step the derivation d *is transformed into another derivation* $d[\![n]\!]$ *and its endsequent* Π *is thereby modified in the following way:*
At most one reduction step $\mathsf{tp}(d)$ *is carried out on the sequent. It may thus happen that an endsequent remains entirely unchanged.*
In this case we set $\mathsf{tp}(d) := \mathsf{Rep}$ with $\mathsf{Rep}(\Pi, n) := \Pi$.

Theorem 3. $d \vdash \Pi \ \Rightarrow\ d[\![n]\!] \vdash \mathsf{tp}(d)(\Pi, n)$.

Inference symbols

R_B, L^k_B (reduction steps on sequents),
Rep, Ax_0.

syntactic variable for inference symbols: $\mathcal{I}$

Definition.

$\mathsf{Ax}_0 \rhd \Pi \ :\Leftrightarrow\ \Pi$ has endform

$\mathsf{Rep} \rhd \Pi \ :\Leftrightarrow\ 0 = 0.$

$\mathsf{Rep}(\Pi, n) := \Pi$

Remark. $\mathcal{I} \rhd \emptyset \to \bot \ \Rightarrow\ \mathcal{I} = \mathsf{Rep}$.

Theorem 3. If $d \vdash \Pi$ then

(i) $\mathsf{tp}(d) \rhd \Pi$;

(ii) $\mathsf{Ax}_0 \not\rhd \Pi \;\Rightarrow\; d[n] \vdash \mathsf{tp}(d)(\Pi, n) \;\; (\forall n)$.

Corollary.

$d \vdash \emptyset{\to}\bot \;\Rightarrow\; d[0] \vdash \emptyset{\to}\bot$.

Proof:

$d \vdash \emptyset{\to}\bot \overset{\text{(i)+Remark}}{\Rightarrow}$

$d \vdash \emptyset{\to}\bot \;\&\; \mathsf{tp}(d) = \mathsf{Rep} \overset{\text{(ii)}}{\Rightarrow}$

$d[0] \vdash \mathsf{Rep}(\emptyset{\to}\bot, 0) = \emptyset{\to}\bot$.

Definition of $\mathsf{tp}(d)$ and $d[n]$

Simultaneously one proves: $d \vdash \Pi \;\Rightarrow\; \mathsf{tp}(d) \rhd \Pi$.

5. $d = \mathbf{K}^r_\Pi d_0 \dots d_l \vdash \Pi$ with $d_i \vdash \Pi_i \;\; (i \le l)$.

Let $j_0 := \min\{j \le l : \mathsf{R}(\Pi_j) = \mathsf{R}(\Pi)\}$.

5.1. d **critical**, i.e. $\forall i \le j_0(\mathsf{tp}(d_i) \not\rhd \Pi)$.

By IH $\forall i \le j_0(\mathsf{tp}(d_i) \rhd \Pi_i)$. It follows that there exist $i < j \le j_0$ and B, k s.t. $\mathsf{tp}(d_i) = \mathsf{R}_B \;\&\; \mathsf{tp}(d_j) = \mathsf{L}^k_B$.

$\mathsf{tp}(d) := \mathsf{Rep}$ and $d[0] := \mathbf{K}^{r-1}_\Pi d\{0\} d\{1\}$ with

$$d\{0\} := \mathbf{K}^r_{\mathsf{tp}(d_i)(\Pi,k)} d_0 \dots d_{i-1} d_i[k] d_{i+1} \dots d_l,$$

$$d\{1\} := \mathbf{K}^r_{\mathsf{tp}(d_j)(\Pi,0)} d_0 \dots d_{j-1} d_j[0] d_{j+1} \dots d_l.$$

5.2. d **not critical**:

Let i be minimal such that $\mathsf{tp}(d_i) \rhd \Pi$.

5.2.1. d_i critical: $\mathsf{tp}(d) := \mathsf{Rep}$ and

$d[0] := \mathbf{K}^{r'}_\Pi d_0 \dots d_{i-1} d_i\{0\} d_i\{1\} d_{i+1} \dots d_l$

5.2.2. d_i not critical:

$\mathsf{tp}(d) := \mathsf{tp}(d_i)$,

$d[n] := \mathbf{K}^r_{\mathsf{tp}(d_i)(\Pi,n)} d_0 \dots d_{i-1} d_i[n] d_{i+1} \dots d_l$.

§8 Transition to multisuccedent seqents (LK)

$\Pi = \Gamma{\to}\Delta$,

$\mathsf{L}(\Pi) := \Gamma$, $\mathsf{R}(\Pi) := \Delta$.

$A, \Pi := A, \Gamma \to \Delta := (\{A\} \cup \Gamma) \to \Delta$ (as before)

$\Pi, A := \Gamma \to \Delta, A := \Gamma \to (\Delta \cup \{A\})$

Kettenschluß: $\dfrac{\Pi_0 \;\; \dots \;\; \Pi_l}{\Pi}$ is a chain inference of degree r if Π can be derived from $\Pi_0, \dots, \Pi_l$ by a finite number of cuts of degree$\le r$.

Definition of $\mathsf{tp}(d)$ and $d[n]$

5. $d = \mathbf{K}^r_\Pi d_0 \dots d_l \vdash \Pi$ with $d_i \vdash \Pi_i \;\; (i \le l)$.

Let $j_0 := l$. Then literally as before.

Lemma (*Existence of a suitable cut*)

If $\dfrac{\Pi_0 \;\; \dots \;\; \Pi_l}{\Pi}$ is a chain inference of degree r, and if $\forall i \le l(\,\mathcal{I}_i \rhd \Pi_i \;\&\; \mathcal{I}_i \not\rhd \Pi\,)$ then

$\exists i, j \le l \exists k, B(\,\mathcal{I}_i = \mathsf{R}_B \;\&\; \mathcal{I}_j = \mathsf{L}^k_B \;\&\; \mathrm{rk}(B) \le r\,)$.