Intuitionistic Decision Procedures since Gentzen

Advances in Proof Theory
(The Jägerfest)
(Annual Meeting of the Swiss Society for Logic and Philosophy of Science)
Bern, **Friday** December **13**, 20**13**
$\sim$**13**.00 hrs (St Andrews Mean Time)

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December 13, 2013

1 Introduction

Our focus is on calculi and procedures that can be understood in relation to traditional proof theory; thus we tend (despite their importance) to avoid implementation issues (e.g. Weich's use [57] of AVL trees rather than lists, Larchay and Galmiche's structure sharing techniques [34], Goubault-Larrecq's (and Goré and Thomson's) binary decision diagrams [25, 24] and Wallen's prefix unification [56]) in favour of relatively simple calculi where questions such as contraction and cut admissibility can be raised and, ideally by syntactic methods, answered. Nor do we address the first-order case, for which see Schütte [49], Franzén et al [46] and Otten [43]. For implementations see Otten's ILTP website [44].

We are particularly interested in questions of **termination** (hence decidability), **bicompleteness** (extractability of models from failed proof searches) and **determinism** (avoidance of backtracking).

We include a short discussion of labelled calculi; concerning termination therein, we refer to some recent literature by Garg et al [21] and by Schmidt et al [48]. Some 2007 work of Antonsen and Waaler [2] is also relevant.

2013 being the 25th anniversary of Hudelmaier's rediscovery [29] of Vorob'ev's calculus (now called **G4ip**), we pay special attention to that calculus.

2 Gentzen's Calculus, LJ

Gentzen [22] solved (by 1935) the decision problem for **Int** with a calculus **LJ**, in which the antecedent of each sequent is a list of formulae and the succedent either empty or a single formula. Since lists rather than sets are used, and the operational rules act only on the first element of the list, rules of Exchange, Contraction and Thinning are required. The rules for conjunction and disjunction being standard, and wlog intuitionistic negation ($\neg$) being considered as a defined notion, the important rules (for intuitionistic implication) are

$$\frac{\Gamma \Longrightarrow A \quad B, \Delta \Longrightarrow C}{A \to B, \Gamma, \Delta \Longrightarrow C}\ L\to \qquad\qquad \frac{A, \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \to B}\ R\to$$

in the first of which C is perhaps empty.

This is not the best of calculi for solving the decision problem—note especially the context-splitting nature of $L\to$; but, defining a sequent to be *reduced* iff its antecedent contains no more than three occurrences of any formula, and after showing that a derivation of a reduced sequent can be modified into one where all the sequents are reduced, one can see an obvious finiteness argument exploiting the subformula property.

Kosta Došen observed in 1987 in [6] that Gentzen's "three occurrences" can be reduced to "two occurrences". One may observe that B subsumes $A \to B$, so in the rule $L\to$ we may need a copy of $A \to B$ in Γ but we don't need one in Δ.

Gentzen's approach is not a root-first approach but to see what sequents (from the finite range of possibilities) are initial, what can be inferred from them, and so on.

3 Calculi of Ketonen and Kleene, G3i

Ketonen [32] and Kleene [33] observed around 1944 (resp. 1950) that it was better to incorporate structural rules (like Weakening, Contraction and Exchange) into the notation (so Γ is now a multiset or set, of formulae, rather than a list) and/or the operational rules, thus obtaining operational rules such as

$$\frac{A{\to}B, \Gamma \Longrightarrow A \quad B, A{\to}B, \Gamma \Longrightarrow C}{A{\to}B, \Gamma \Longrightarrow C}\ L{\to} \qquad \frac{A, \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A{\to}B}\ R{\to}$$

and the convention that two sequents are "cognate" (and thus are interchangeable) iff exactly the same formulae appear in the antecedents (regardless of number and order) and they have the same succedent.

Note that $A{\to}B$ can be omitted from the second premiss of $L{\to}$ (since it is subsumed by B), but not from the first, lest completeness be lost.

This now allows a "root-first" approach.

4 Maehara's Calculus, m-G3i

Maehara [37] introduced an important variant of Kleene's calculus: succedents can now be arbitrary (finite) sets Δ of formulae rather than just empty or singular. The rules for implication are then

$$\frac{A \to B, \Gamma \Longrightarrow A, \Delta \quad B, A \to B, \Gamma \Longrightarrow \Delta}{A \to B, \Gamma \Longrightarrow \Delta} L\to \qquad \frac{A, \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A \to B, \Delta} R\to$$

which have the virtue that $L\to$ is invertible and that all the non-determinism in root-first search pertains to the $R\to$ rule and the choice of implicational formula $A \to B$ in the succedent for analysis. (The $R\vee$ rule is also made invertible.)

Perhaps more importantly, proofs in this system can be much smaller than those in the single-succedent calculus: see Egly and Schmitt [15] for details.

Approximately this calculus is used as a basis in tableau theorem proving; one advantage is that counter-models can be extracted from failed searches. (Note that the rule $R\vee$ is classical here but not in **G3i**.) In other words, the calculus is *bicomplete*.

The same calculus (presented as a tableau calculus) appears in Fitting's thesis [18], attributed to Beth [3], and in his book [19].

Fitting's notion of "tableau" is a finite sequence of configurations, each obtained from its predecessor by applying a rule; each *configuration* is a finite collection of problems (each of which has to be solved for the configuration to be *closed*).

Backtracking (because of the rule $R\to$) is not made explicit; conjunctive branching is handled by adding an extra problem (sequent, i.e. "set of signed formulas").

Termination is assured by the subformula property, i.e. some form of loop-checking is required.

An interesting variation is the calculus **GHPC** of Dragalin [7]; by omitting Δ, this has a non-invertible $L\to$ rule, incorporating a form of focusing useful in the proof theory of the multi-succedent **m-G4ip**.

5 Vorob'ev's Calculus, G4ip

N. N. Vorob'ev introduced (c. 1950) in papers [54], [55] an important calculus now known as **G4ip**. Others (Hudelmaier [29, 30, 31], RD [8]) rediscovered (and refined) the same calculus some 40 years later. See also Lincoln et al [35].

The key idea is to replace, in a single succedent calculus **G3ip**, the left rule for implication by four rules, according to the form of the implication's antecedent, exploiting the equivalences $(C \vee D) \to B \equiv (C \to B) \wedge (D \to B)$, $(C \wedge D) \to B \equiv C \to (D \to B)$, $C \wedge ((C \to D) \to B) \equiv C \wedge (D \to B)$ and $P \wedge (P \to B) \equiv P \wedge B$ to reduce the complexity (in a carefully measured sense) of the formula and a bit of proof theory to show completeness.

The effect is that depth-first proof search terminates, i.e. root-first application of inference rules decreases the sequent's "size" rather than allowing it to oscillate up and down without termination. A measure of "size" (due to Hudelmaier) can be found in [52].

The rules for implication on the left are thus as follows:

$$\frac{\Gamma, P, B \Longrightarrow E}{\Gamma, P, P \to B \Longrightarrow E}\ L0\!\to \qquad \frac{\Gamma, C \to (D \to B) \Longrightarrow E}{\Gamma, (C \wedge D) \to B \Longrightarrow E}\ L\!\wedge\!\to$$

$$\frac{\Gamma, C \to B, D \to B \Longrightarrow E}{\Gamma, (C \vee D) \to B \Longrightarrow E}\ L\!\vee\!\to \qquad \frac{\Gamma, C, D \to B \Longrightarrow D \quad \Gamma, B \Longrightarrow E}{\Gamma, (C \to D) \to B \Longrightarrow E}\ L\!\to\!\to$$

of which each but the last is invertible.

6 Hudelmaier's refinements of Vorob'ev's Calculus

First appearance of Hudelmaier's rediscovery of Vorob'ev's work is in [29], i.e. in 1988.

Novelty (apart from some different proof methods) w.r.t. **G4ip** is to ensure proofs are of linear rather than exponential depth, by use of fresh proposition variables in the cases ($L{\vee}{\to}$ and $L{\to}{\to}$) where a non-atomic subformula (B, resp. D) from the conclusion is duplicated into a premiss. See Hudelmaier's [30] and [31].

This shows that the decision problem is in $O(n\ log\ n)$-SPACE. (In 1977 Ladner showed **S4** to be in PSPACE, and hence so is **Int**; Statman showed **Int** to be P-SPACE-hard [50].)

7 RD's refinements [8] of Vorob'ev's Calculus

Novelty (apart from different proof methods) is to have (in addition to the single succedent calculus **G4ip**) a multi-succedent calculus **m-G4ip**, closer to tableau methods used in implementations and allowing extraction of a counter-model from a failed proof search [45] (joint work with Pinto) . For the multi-succedent version, just replace each succedent formula E by Δ.

Can be combined with Hudelmaier's depth-reduction techniques.

Various refinements of the multi-succedent version have been developed and implemented by a group in Milan (Avellone, Ferrari, Fiorentino, Fiorino, Miglioli†, Moscato and Ornaghi); one of the most recent papers is [17]. Their proof methods are almost entirely semantic.

8 Proof theory of Vorob'ev's Calculus

Vorob'ev's proof of completeness of the calculus rests on a lemma now seen as the completeness of a single-succedent focused calculus **LJQ'**: see RD and Lengrand [10] for details, and its extension to a multi-succedent focused calculus **LJQ*** (a variant of a calculus in Herbelin's thesis [26]).

Root-first proof search in **LJQ'** occasionally focuses on the succedent and analyses it until either it is atomic or the rule $R{\rightarrow}$ is used; in particular, the $L{\rightarrow}$ rule requires (in the first premiss, but not in the second) a focus on the succedent.

The completeness of this approach is a useful fact, exploited not just in Vorobev's [54, 55] but also in Hudelmaier's [31] (in which it is mentioned as "folklore").

RD and Negri [11] give a direct proof of completeness (w.r.t. an axiomatic presentation, via Cut-admissibility, rather than w.r.t. semantics), showing that Contraction is admissible in **G4ip** and hence (with explicit cut reduction steps) that Cut is admissible.

This approach generalises to the multi-succedent case, and even shows the completeness of a first-order version (without, alas, the depth-boundedness . . .).

RD , Kesner and Lengrand [9] show (for the implicational fragment **G4ip**$^{\rightarrow}$ only) how to make the cut reduction system strongly normalising.

9 Weich's thesis

Weich [57, 58] made several excellent contributions:

1. Verified constructive completeness proofs, in *MINLOG* and in *Coq*, from which Scheme or OCaml programs may be extracted;

2. Pruning of the search by use of counter-models generated earlier in the search ("an improvement both astonishing and significant");

3. A "conditional normal form" for formulae, obtained by pre-processing: essentially, $A \to B$ where A is a conjunction of atoms and B is one of $\bot, P, Q \vee R, (Q \to R) \to \bot, (Q \to R) \to S$. This reduces some of the run-time expansions that are otherwise repeated in different branches of the search. (P, Q, R, S indicate atoms.)

10 Easy optimisations

1. Once the succedent is empty (or just $\bot$), one can revert to classical logic.

2. Search can be pruned if a new subproblem (arising from choice of instance of non-invertible rule) isn't solvable classically.

3. If using a single-succedent calculus, and with an atomic succedent P, one may restrict analysis of antecedent implicational formulae to those that contain P strictly positively. Thus, the sequent $(p \to s) \to t, (c \to p) \to b \Longrightarrow p$ cannot be reduced (but would be reduced if we had $p = t$). One can see this as a form of "goal-directedness".

4. When a problem is analysed into two subproblems, and the first is solved, one may use [57] information from it in the second; e.g. the two rules (for multi-succedent or single-succedent calculi, resp.)

$$\frac{A, \Gamma \Longrightarrow \Delta \quad B, \Gamma \Longrightarrow A, \Delta}{A \vee B, \Gamma \Longrightarrow \Delta} L\vee' \qquad \frac{A, \Gamma \Longrightarrow G \quad B, A \to G, \Gamma \Longrightarrow G}{A \vee B, \Gamma \Longrightarrow G} L\vee''.$$

5. "Simplification": once an atom p is added to the antecedent, all formulae in the sequent are simplified by putting $p = \top$ and reducing (e.g. with $\top \wedge A \equiv A$).

6. The same works if a negated atom $\neg p$ is added to the antecedent; the sequent is simplified by replacing p throughout by $\bot$ (and simplifying accordingly, e.g. with $\bot \vee A \equiv A$).

7. Several other easy optimisations are to be found in Franzén's [20], Ferrari et al's [17] and Weich's [57].

11 Mints' classification

Mints [40] gave a convenient classification of subclasses of **Int**, and their complexity. Let $|S|$ be the formula equivalent of a sequent S. By introduction of new variables (following Skolem 1920 and Wajsberg 1938), one can in linear time replace a formula A by a sequent S_A (in an extension of the language with the new variables) so that A is provable iff $|S_A|$ is provable, where the succedent of S_A is atomic and the antecedent consists solely of formulae that (with P, Q, R atomic) are either (0) **atoms** P, (1) **negations** $\neg P$, (2) **implications** $P \to Q$, (3) **binary implications** $P \to (Q \to R)$, (4) **nested implications** $(P \to Q) \to R$, (5) **implied disjunctions** $P \to (Q \vee R)$, (6) **negative implications** $P \to (\neg Q)$ and (7) **converse negative implications** $\neg Q \to P$. Thus, it suffices to consider only sequents where the antecedent X consists of formulae of these eight types (and the succedent is atomic).

According to the types of formulae used in X, one has complexity results: if all formulae of X are of type 2, 3 or 4 we talk of the class [2,3,4], and similarly for other classes. One then has that the class [2,3,4] (and any superclass) is PSPACE-complete; the class [1,2,5,6] is NP-complete (and any superclass is NP-hard); but the class [0,1,2,3,6] (and any subclass) is in LIN; the class [0,1,2,4,5,7] (and any subclass) is in P; and so on.

From the perspective of **G4ip**, the difficulty of proof search is dealing with nested implications, i.e. formulae of type (4). So the surprise is that (provided we exclude formulae of type (3) and (6)) while allowing formulae of type (4) and their variant (7), the decision problem is in P. This is achieved using a resolution method [39], a variant of the familiar "forward chaining" method that disposes linearly of [0,1,2,3,6]. Tammet [51] implemented this method: but the verdict [44] by the ILTP website authors is "Prover seems to be incorrect".

12 Ensuring the subformula property

G4ip lacks the subformula property, and has been criticised by some for this failing, apparently on philosophical grounds. Despite a strong feeling that it doesn't matter (because it is still analytic in a weak but adequate sense), we consider henceforth some further approaches that ensure that proofs have the subformula property:

1. Underwood's calculus
2. Implication-locking (Franzén)
3. Loop-checking (two approaches)
4. The calculus **LJPm*** of Mints
5. The calculi **IG**^{r} and **SIC** of Corsi and Tassi
6. The calculus **LSJ** of Ferrari, Fiorentino and Fiorino
7. The calculus **GLJ** of RD (unpublished)

13 Underwood's Calculus

Underwood [53] in 1990 gave a constructive completeness proof for a calculus presented rather in terms of Kripke semantics than proof theory.

As reconstructed by Weich [57], this is as follows, with antecedents and succedents regarded as sets: rules for conjunction and disjunction are rather standard, with provisos about not being used if they fail to add a new formula to one of the sets.

Rules for implication are thus:

$$\frac{A{\to}B, \Gamma \Longrightarrow A, \Delta \quad B, A{\to}B, \Gamma \Longrightarrow \Delta}{A{\to}B, \Gamma \Longrightarrow \Delta}\ L{\to} \qquad \frac{A, \Gamma \Longrightarrow B, A{\to}B, \Delta}{A, \Gamma \Longrightarrow A{\to}B, \Delta}\ RSimp \qquad \frac{A, \Gamma \Longrightarrow B}{\Gamma \Longrightarrow A{\to}B, \Delta}\ R{\to}$$

provided that, in $L{\to}$, $A \notin \Delta$ and $B \notin \Gamma$, and, in $RSimp$, $B \notin \Delta$, and, in $R{\to}$, $A \notin \Gamma$.

The lengths of branches in this calculus are bounded by the square of the number of subformulae of the sequent to be proved, hence termination. No loop-checking is required: just the check of the various provisos.

The calculus is the basis for the extraction of an algorithm by Caldwell [4].

14 Implication-locking (Franzén's approach)

This work by Franzén [20, 46] uses the notion of covering: Γ *covers* A if $A \in \Gamma$, or $A \equiv B \wedge C$ and Γ covers both B and C, or $A \equiv B \vee C$ and Γ covers one of B and C, or $A \equiv B \to C$ and Γ covers C.

The rule $R\to$ is then specialised to the two cases: the usual one (a *transfer instance*) if Γ does not cover the antecedent A of the principal formula, and the special one (from $\Gamma \Longrightarrow B$ infer $\Gamma \Longrightarrow A \to B$ when Γ covers A).

There is then the restriction that, on each branch, every two instances of $L\to$ must be separated by a transfer instance of $R\to$. In other words, implications are locked until released by a transfer. This is enough to ensure termination.

15 Loop-checking (the Bern approach)

See [27] (by Heuerding et al, 1996) for details.

With a little loss of generality, we ignore disjunction and absurdity. We may therefore restrict $L\rightarrow$ to cases where the succedent formula is an atom.

Left rules are *cumulative*, i.e. the principal formula is duplicated to the premiss. So a loop can only occur during a phase when nothing new is added to the antecedent, and in the succedent a formula appears and later (i.e. higher up the proof branch) appears again. Wlog one can restrict to the case where this formula is an atom.

Sequents now contain an extra component, the *history* (which is the set of such atoms).

So, if (as one moves root-first) a new formula is added to the antecedent, the history at the premiss is emptied. The atomic succedent P is added to the history H when the left rule for implication is used (unless already $P \in H$, in which case the branch is blocked).

16 Loop-checking (the St Andrews approach)

Sequents again contain an extra component, the *history*. Howe [28] presented a variation of the Bern approach; in Howe's variation, loops are found earlier at the cost of some extra data storage; in some cases this dramatically cuts the search time, but in general makes it slightly slower.

17 System LJpm* of Mints

Mints' inference rules [41] operate on "tableaux", i.e. lists $\mathcal{T}$ of multi-succedent sequents (the *components* of $\mathcal{T}$). Conjunctive branching replaces one tableau by two, whereas disjunctive branching extends a tableau; a "proof" is a tree each leaf of which is initial, where a tableau is *initial* iff one of its components is (in one of the conventional ways) initial. Use of tableaux rather than just of sequents avoids backtracking at the meta-level: all the inference rules are invertible.

Here for example are the rules for implication (using ";" for the "append" operation on lists, where [41] uses a "$\star$"; and, for emphasis, we have parenthesised components):

$$\frac{\mathcal{T};\ (A\to B,\Gamma \Longrightarrow A,\Delta);\ \mathcal{T}' \quad \mathcal{T};(B,\Gamma \Longrightarrow \Delta);\mathcal{T}'}{\mathcal{T};\ (A\to B,\Gamma \Longrightarrow \Delta);\ \mathcal{T}'}\ L\to \qquad \frac{\mathcal{T};\ \ (\Gamma \Longrightarrow \Delta, A\to B);\ \ (\Gamma, A \Longrightarrow B);\ \ \mathcal{T}'}{\mathcal{T};\ (\Gamma \Longrightarrow \Delta, A\to B);\ \mathcal{T}'}\ R\to$$

in which note the conjunctive branching in the first and the disjunctive branching (by extension of the tableau) in the second.

Termination is achieved by fixing all the rules so that the principal formula is always (except for $R\to$) duplicated into the premisses, and search along a branch is terminated when one reaches a tableau to "which no rule can be meaningfully applied". To make this precise, one defines that $\Gamma \Longrightarrow \Delta$ *subsumes* $\Gamma' \Longrightarrow \Delta'$ iff $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$ (as sets of formulae); then one forbids any tableau extension step if some new sequent subsumes some component of some tableau lower down the tree, i.e. loops must be avoided. A finiteness argument then shows that this ensures termination.

18 System $\mathbf{IG}^r$ of Corsi and Tassi [5]

We present the system $\mathbf{IG}^r$ of Corsi and Tassi 2007 (implicational part: the other parts present no difficulties). Initial sequents are, as usual, those with an atom on both left and right.) Its main features are (a) that it is depth-bounded (b) that it has the subformula property and (c) bicompleteness. The superfix r stands for a regularity condition, enforced by the use of $\mathcal{B}$ and $\mathcal{H}$. The rules $AF\neg$ and $AF\!\to$ implement what the authors call an "A Fortiori" condition, seen most clearly in the second of these two rules.

$\mathcal{B}$ is for a list of *blocked* formulae, i.e. the list of all formulae on the path from here to root that have been principal for $L\neg$ or $L\!\to$. Unless they are unblocked, they should not be reused.

$\mathcal{H}$ is for a *History* list, i.e. the list of all formulae on the path from here to root that have been principal for $R\neg$ or $R\!\to$.

$\mathcal{B}$ is cleared whenever (as one proceeds up such a path) there is a use of $R\!\to$ or $R\neg$. $\mathcal{H}$ is never cleared.

$$\frac{\neg A, \Gamma \underset{\neg A, \mathcal{B}\,;\,\mathcal{H}}{\Longrightarrow} A, \Delta}{\neg A, \Gamma \underset{\mathcal{B}\,;\,\mathcal{H}}{\Longrightarrow} \Delta}\ L\neg \quad (\text{if } \neg A \notin \mathcal{B}) \qquad \frac{A\to B, \Gamma \underset{A\to B, \mathcal{B}\,;\,\mathcal{H}}{\Longrightarrow} A, \Delta \quad B, \Gamma \underset{A\to B, \mathcal{B}\,;\,\mathcal{H}}{\Longrightarrow} \Delta}{A\to B, \Gamma \underset{\mathcal{B}\,;\,\mathcal{H}}{\Longrightarrow} \Delta}\ L\!\to \quad (\text{if } A\to B \notin \mathcal{B})$$

$$\frac{A, \Gamma \underset{[]\,;\,\neg A, \mathcal{H}}{\Longrightarrow}}{\Gamma \underset{\mathcal{B}\,;\,\mathcal{H}}{\Longrightarrow} \neg A, \Delta}\ R\neg \quad (\text{if } \neg A \notin \mathcal{H}) \qquad \frac{A, \Gamma \underset{[]\,;\,A\to B, \mathcal{H}}{\Longrightarrow} B}{\Gamma \underset{\mathcal{B}\,;\,\mathcal{H}}{\Longrightarrow} A\to B, \Delta}\ R\!\to \quad (\text{if } A\to B \notin \mathcal{H})$$

$$\frac{\Gamma \underset{\mathcal{B}\,;\,\mathcal{H}}{\Longrightarrow} \Delta}{\Gamma \underset{\mathcal{B}\,;\,\mathcal{H}}{\Longrightarrow} \neg A, \Delta}\ AF\neg \quad (\text{if } \neg A \in \mathcal{H}) \qquad \frac{\Gamma \underset{\mathcal{B}\,;\,\mathcal{H}}{\Longrightarrow} B, \Delta}{\Gamma \underset{\mathcal{B}\,;\,\mathcal{H}}{\Longrightarrow} A\to B, \Delta}\ AF\!\to \quad (\text{if } A\to B \in \mathcal{H})$$

Here are proofs (in two styles, each using a one-dimensional layout $\mathcal{B}; \mathcal{H} : \Gamma \Longrightarrow \Delta$ for each sequent) of the formula that is the type of the S combinator:

$$\dfrac{\dfrac{}{[p\to q]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [p, p\to q, p\to q\to r] \Longrightarrow [p, r]}Ax \quad \dfrac{\dfrac{}{[p\to q\to r, p\to q]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [q, p, p\to q\to r] \Longrightarrow [p, r]}Ax \quad \dfrac{\dfrac{}{[q\to r, p\to q\to r, p\to q]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [q\to r, q, p] \Longrightarrow [q, r]}Ax \quad \dfrac{}{[q\to r, p\to q\to r, p\to q]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [r, q, p] \Longrightarrow [r]}Ax}{[p\to q\to r, p\to q]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [q\to r, q, p] \Longrightarrow [r]}L\to}{[p\to q]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [q, p, p\to q\to r] \Longrightarrow [r]}L\to}{\dfrac{\dfrac{\dfrac{[]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [p, p\to q, p\to q\to r] \Longrightarrow [r]}{[]; [(p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [p\to q, p\to q\to r] \Longrightarrow [p\to r]}R\to}{[]; [(p\to q\to r)\to(p\to q)\to p\to r] : [p\to q\to r] \Longrightarrow [(p\to q)\to p\to r]}R\to}{[]; [] : [] \Longrightarrow [(p\to q\to r)\to(p\to q)\to p\to r]}R\to}L\to$$

$$[p\to q]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [p, p\to q, p\to q\to r] \Longrightarrow [p, r] \tag{1}$$
$$[p\to q\to r, p\to q]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [q, p, p\to q\to r] \Longrightarrow [p, r] \tag{2}$$
$$[q\to r, p\to q\to r, p\to q]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [q\to r, q, p] \Longrightarrow [q, r] \tag{3}$$
$$[q\to r, p\to q\to r, p\to q]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [r, q, p] \Longrightarrow [r] \tag{4}$$
$$[p\to q\to r, p\to q]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [q\to r, q, p] \Longrightarrow [r] \qquad \text{By} L\to \ \text{from } 3, 4 \tag{5}$$
$$[p\to q]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [q, p, p\to q\to r] \Longrightarrow [r] \qquad \text{By} L\to \ \text{from } 2, 5 \tag{6}$$
$$[]; [p\to r, (p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [p, p\to q, p\to q\to r] \Longrightarrow [r] \qquad \text{By} L\to \ \text{from } 1, 6 \tag{7}$$
$$[]; [(p\to q)\to p\to r, (p\to q\to r)\to(p\to q)\to p\to r] : [p\to q, p\to q\to r] \Longrightarrow [p\to r] \qquad \text{By} R\to \ \text{from } 7 \tag{8}$$
$$[]; [(p\to q\to r)\to(p\to q)\to p\to r] : [p\to q\to r] \Longrightarrow [(p\to q)\to p\to r] \qquad \text{By} R\to \ \text{from } 8 \tag{9}$$
$$[]; [] : [] \Longrightarrow [(p\to q\to r)\to(p\to q)\to p\to r] \qquad \text{By} R\to \ \text{from } 9 \tag{10}$$

with shape

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{1} \quad \dfrac{\overline{2} \quad \dfrac{\overline{3} \quad \overline{4}}{5} L\to}{6} L\to}{7} L\to}{8} R\to}{9} R\to}{10} R\to$$

A proof in the **G4ip** calculus looks (in tree form) much the same but with many fewer formulae at each node and without branching, using the rule $L0\to$ rather than $L\to$.

19 The calculus SIC of Corsi and Tassi

SIC is a variant of the system $\mathbf{IG}^r$ in the same paper [5]; the essential difference is that backtracking (because of disjunctive branching) is incorporated into the calculus, and thus each node of the tree is a stack of ordinary sequents rather than just one such sequent. This is very similar to Mints' notion of tableau.

Sequents are pushed onto the stack to indicate all the alternative possibilities (according to the different implicational succedent formulae); as they are tried and found unsolvable, they are popped, and failure occurs when the stack is empty.

The goal is thus achieved that all use of "global metarules" is thus replaced by use of "local metarules", incorporated into the rules of the calculus.

20 The calculus LSJ of Ferrari, Fiorentino and Fiorino

Sequents are [16] of the form $\Theta; \Gamma \Longrightarrow \Delta$, the components being sets (of formulae) rather than multisets. The semantics (using $<$ for $\leq$ without equality, and only finite models) of such a sequent is that $(K, V, w) \Vdash \Theta; \Gamma \Longrightarrow \Delta$ iff whenever (1) for every $H \in \Theta$ and $w' \in K$ with $w < w'$, one has $(K, V, w') \Vdash H$ and (2) for every $G \in \Gamma$, one has $(K, V, w) \Vdash G$ then (3) for some $D \in \Delta$ one has $(K, V, w) \Vdash D$. Negation is defined as usual. The rules are

$$\frac{}{\Theta; \bot, \Gamma \Longrightarrow \Delta}\ L\bot \qquad \frac{}{\Theta; A, \Gamma \Longrightarrow A, \Delta}\ Id$$

$$\frac{\Theta; A, B, \Gamma \Longrightarrow \Delta}{\Theta; A \wedge B, \Gamma \Longrightarrow \Delta}\ L\wedge \qquad \frac{\Theta; \Gamma \Longrightarrow A, \Delta \quad \Theta; \Gamma \Longrightarrow B, \Delta}{\Theta; \Gamma \Longrightarrow A \wedge B, \Delta}\ R\wedge$$

$$\frac{\Theta; A, \Gamma \Longrightarrow \Delta \quad \Theta; B, \Gamma \Longrightarrow \Delta}{\Theta; A \vee B, \Gamma \Longrightarrow \Delta}\ L\vee \qquad \frac{\Theta; \Gamma \Longrightarrow A, B, \Delta}{\Theta; \Gamma \Longrightarrow A \vee B, \Delta}\ R\vee$$

$$\frac{\Theta; B, \Gamma \Longrightarrow \Delta \quad B, \Theta; \Gamma \Longrightarrow A, \Delta \quad B; \Theta, \Gamma \Longrightarrow A}{\Theta; A \to B, \Gamma \Longrightarrow \Delta}\ L\to \qquad \frac{\Theta; A, \Gamma \Longrightarrow B, \Delta \quad []; A, \Theta, \Gamma \Longrightarrow B}{\Theta; \Gamma \Longrightarrow A \to B, \Delta}\ R\to$$

A syntactic proof of cut-admissibility for this calculus seems difficult; a semantic proof is in [16].

Using our own implementation of **LSJ**, with Prolog cuts to prune the search space wherever seemed appropriate, the first (indeed, only) proof we found of the formula that is the type of the S combinator is 87 lines long. It is possible that, with differently placed cuts in the implementation, a shorter proof would be found.

An associated calculus, building on the approach of [45], gives bicompleteness.

21 The calculus GLJ of RD (unpublished)

We consider Sambin and Valentini's system **GLS'** from [47] for the provability logic **GL** (implicational and modal part: the other parts are standard). All rules (except RR) are invertible (they call this *doubly sound*). Initial sequents are, as is almost usual, those with a formula on both left and right (or with $\bot$ on the left).

$$\frac{\Gamma \Longrightarrow A, \Delta \quad \Gamma, B \Longrightarrow \Delta}{\Gamma, A \supset B \Longrightarrow \Delta} L\supset \quad \frac{\Gamma, A \Longrightarrow B, \Delta}{\Gamma \Longrightarrow A \supset B, \Delta} R\supset$$

$$\frac{\Gamma, \Box\Gamma, \Box D \Longrightarrow D}{\Pi, \Box\Gamma \Longrightarrow \Box\Delta, \Sigma} RR \quad \text{where } D \in \Delta$$

RR (in which Π and Σ are disjoint sets of atoms; likewise, the sets Γ and Δ are w.lo.g. disjoint) has the property that if the conclusion is valid then, for some $D \in \Delta$, the corresponding premiss is valid. (The authors also call this "doubly sound".) When we see $A \supset B, \Delta$ in a conclusion, it is implicit that $A \supset B \notin \Delta$; similarly with Γ rather than Δ, and for other connectives.

Root-first proof search in **GLS'** terminates. The argument is as follows (from [47]). First, used root-first, every rule other than RR reduces the number of connectives. Second, as we proceed up a branch, the set of boxed formulae in the antecedent occasionally expands but never shrinks: thus, if a sequent $\Pi, \Box\Gamma \Longrightarrow \Box\Delta, \Sigma$ is a conclusion of RR, the antecedent of every sequent above it will contain a formula $\Box D$, with $D \in \Delta$, and the antecedent of every sequent at or below it cannot contain such a formula (since search stops at initial sequents). So all the sequents in a branch are different. By the subformula property their number is finite, so search along any branch terminates.

22 GLJ (continued)

Consider the standard embedding $\cdot^{\boxdot}$ of **Int** into **GL**, in which notice the careful distinction between classical and intuitionistic implication, $A \supset B$ and $A \to B$, like that between classical and intuitionistic negation, $\sim A$ and $\neg A$:

$$\begin{array}{rcl@{\qquad\qquad}rcl}
\bot^{\boxdot} & := & \bot & P^{\boxdot} & := & P \wedge \Box P \\
(A \wedge B)^{\boxdot} & := & A^{\boxdot} \wedge B^{\boxdot} & (A \vee B)^{\boxdot} & := & A^{\boxdot} \vee B^{\boxdot} \\
(\neg A)^{\boxdot} & := & \sim A^{\boxdot} \wedge \Box(\sim A^{\boxdot}) & (A \to B)^{\boxdot} & := & (A^{\boxdot} \supset B^{\boxdot}) \wedge \Box(A^{\boxdot} \supset B^{\boxdot})
\end{array}$$

The interpretations of the intuitionistic implication rules are then

$$\frac{\Gamma^{\boxdot}, (A \to B)^{\boxdot} \Longrightarrow A^{\boxdot}, \Delta^{\boxdot} \quad \Gamma^{\boxdot}, B^{\boxdot} \Longrightarrow \Delta^{\boxdot}}{\Gamma^{\boxdot}, (A \to B)^{\boxdot} \Longrightarrow \Delta^{\boxdot}} \; L\!\to^{\boxdot} \qquad\qquad \frac{\Gamma^{\boxdot}, A^{\boxdot} \Longrightarrow B^{\boxdot}}{\Gamma^{\boxdot} \Longrightarrow (A \to B)^{\boxdot}, \Delta^{\boxdot}} \; R\!\to^{\boxdot}$$

and these need to be justified as sound rules in **GLS'** (in which Weakening and Contraction are known to be admissible). For an arbitrary intuitionistic implication $A \to B$, let $(A \to B)^* =_{def} A^{\boxdot} \supset B^{\boxdot}$.

For $L\!\to^{\boxdot}$, from its two premisses we obtain, by Left-Weakening the second with $(A \to B)^{\boxdot}$ and then $L\supset$, that $\Gamma^{\boxdot}, (A \to B)^{\boxdot}, A^{\boxdot} \supset B^{\boxdot} \Longrightarrow \Delta^{\boxdot}$; Left-Weaken with $\Box(A^{\boxdot} \supset B^{\boxdot})$, use $L\wedge$ to replace the formulae $A^{\boxdot} \supset B^{\boxdot}$ and $\Box(A^{\boxdot} \supset B^{\boxdot})$ by $(A \to B)^{\boxdot}$; Contract on $(A \to B)^{\boxdot}$ to obtain $\Gamma^{\boxdot}, (A \to B)^{\boxdot} \Longrightarrow \Delta^{\boxdot}$, as required.

For $R \to {}^{\boxdot}$, let us ignore absurdity, conjunctions and disjunctions (which can always be unpacked using invertible rules of **GLS'**). So w.l.o.g. Γ consists of atoms Π and implications Θ. Let $D =_{def} A^{\boxdot} \supset B^{\boxdot}$. From the premiss of $R\!\to^{\boxdot}$ we obtain by $R\supset$ that $\Gamma^{\boxdot} \Longrightarrow D$. Note that (using invertible rules) $\Gamma^{\boxdot} \equiv \Pi, \Theta^*, \Box(\Pi, \Theta^*)$. Use these inversions and Left-Weaken with $\Box D$; we obtain $\Pi, \Theta, \Box(\Pi, \Theta), \Box D \Longrightarrow D$ from which, by RR, follows $\Box(\Pi, \Theta) \Longrightarrow \Box D$. Left-Weaken this with Π, Θ and use $L\wedge$ several times; we now have just $\Gamma^{\boxdot}$ on the left. An instance of $R\wedge$ (and then Right-Weakening with $\Delta^{\boxdot}$) gets us, as required, $\Gamma^{\boxdot} \Longrightarrow (A \to B)^{\boxdot}, \Delta^{\boxdot}$.

We have thus (to exercise the notation) justified the (well-known) faithfulness of the translation $\cdot^{\boxdot}$ of **Int** into **GL**. It is of greater interest to explain the converse, i.e. the fullness of the embedding, using the quite restrictive rules of **GLS'**.

Instead, we specialise the rules of **GLS'** and show them in the language of **Int**. In the following, Π and Σ are sets of atoms; Θ and Ψ are sets of atoms, classical negations or classical implications, implicitly treated as boxed; Γ and Δ are arbitrary sets of formulae—and all are just formulae of **Int**, apart from the classical negations and implications. P will range over atoms.

Sequents (of the new calculus **GLJ**) are now of the form $\Pi;\Theta;\Gamma \Longrightarrow \Delta;\Psi;\Sigma$. Provability of a formula A will match the derivability of the sequent $[];[];[] \Longrightarrow [A];[];[]$. We use the classical notation $A\supset B$ for implications that are moved by rule $L\!\rightarrow$ from Γ to Θ (or by rule $R\!\rightarrow$ from Δ to Ψ), so that when moved back to Γ by variants of RR they are correctly analysed by $L\supset$.

$$\frac{}{P,\Pi;\ \Theta;\ \Gamma \Longrightarrow \Delta;\ \Psi;\ P,\Sigma}\ Ax1 \qquad \frac{}{\Pi;\ A,\Theta;\ \Gamma \Longrightarrow \Delta;\ A,\Psi;\ \Sigma}\ Ax2$$

$$\frac{}{\Pi;\ \Theta;\ \bot,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ L\bot \qquad \frac{\Pi;\ \Theta;\ \Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ \Gamma \Longrightarrow \bot,\Delta;\ \Psi;\ \Sigma}\ R\bot$$

$$\frac{P,\Pi;\ P,\Theta;\ \Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ P,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ LAt \qquad \frac{\Pi;\ \Theta;\ \Gamma \Longrightarrow \Delta;\ \Psi;\ P,\Sigma \quad \Pi;\ \Theta;\ \Gamma \Longrightarrow \Delta;\ P,\Psi;\ \Sigma}{\Pi;\ \Theta;\ \Gamma \Longrightarrow P,\Delta;\ \Psi;\ \Sigma}\ RAt$$

$$\frac{\Pi;\ \Theta;\ A,B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ A\wedge B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ L\wedge \qquad \frac{\Pi;\ \Theta;\ \Gamma \Longrightarrow A,\Delta;\ \Psi;\ \Sigma \quad \Pi;\ \Theta;\ \Gamma \Longrightarrow B,\Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ \Gamma \Longrightarrow A\wedge B,\Delta;\ \Psi;\ \Sigma}\ R\wedge$$

$$\frac{\Pi;\ \Theta;\ A,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma \quad \Pi;\ \Theta;\ B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ A\vee B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ L\vee \qquad \frac{\Pi;\ \Theta;\ \Gamma \Longrightarrow A,B,\Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ \Gamma \Longrightarrow A\vee B,\Delta;\ \Psi;\ \Sigma}\ R\vee$$

$$\frac{\Pi;\ A\supset B,\Theta;\ \Gamma \Longrightarrow A,\Delta;\ \Psi;\ \Sigma \qquad \Pi;\ A\supset B,\Theta;\ B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ A\!\rightarrow\! B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ L\!\rightarrow$$

$$\frac{\Pi;\ \Theta;\ \Gamma \Longrightarrow A,\Delta;\ \Psi;\ \Sigma \qquad \Pi;\ \Theta;\ B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ A\supset B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ L\supset$$

$$\frac{\Pi;\ \Theta;\ A,\Gamma \Longrightarrow B,\Delta;\ \Psi;\ \Sigma \qquad \Pi;\ \Theta;\ \Gamma \Longrightarrow \Delta;\ A\supset B,\Psi;\ \Sigma}{\Pi;\ \Theta;\ \Gamma \Longrightarrow A\!\rightarrow\! B,\Delta;\ \Psi;\ \Sigma}\ R\!\rightarrow$$

$$\frac{\Pi;\ \sim\!A,\Theta;\ \Gamma \Longrightarrow A,\Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ \neg A,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ L\neg \qquad \frac{\Pi;\ \Theta;\ \Gamma \Longrightarrow A,\Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ \sim\!A,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ L\sim$$

$$\frac{\Pi;\ \Theta;\ A,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma \qquad \Pi;\ \Theta;\ \Gamma \Longrightarrow \Delta;\ \sim\!A,\Psi;\ \Sigma}{\Pi;\ \Theta;\ \Gamma \Longrightarrow \neg A,\Delta;\ \Psi;\ \Sigma}\ R\neg$$

$$\frac{[];\ A\supset B,\Theta;\ A,\Theta \Longrightarrow B;\ [];\ []}{\Pi;\ \Theta;\ [] \Longrightarrow [];\ A\supset B,\Psi;\ \Sigma}\ RR\supset \qquad \frac{[];\ \sim\!A,\Theta;\ A,\Theta \Longrightarrow [];\ [];\ []}{\Pi;\ \Theta;\ [] \Longrightarrow [];\ \sim\!A,\Psi;\ \Sigma}\ RR\sim \qquad \frac{[];\ P,\Theta;\ \Theta \Longrightarrow [];\ [];\ P}{\Pi;\ \Theta;\ [] \Longrightarrow [];\ P,\Psi;\ \Sigma}\ RRAt$$

In the rules $RR\supset$, $RR\sim$ and $RRAt$, it is required that Π and Σ are disjoint and Θ is disjoint from (respectively) $A \supset B, \Psi$, from $\sim A, \Psi$ and from P, Ψ.

This calculus doesn't quite have the subformula property: for example, $L\!\rightarrow$ and $R\!\rightarrow$ turn intuitionistic implications into classical implications. To obtain it, we can either decree that $A\supset B$ is a subformula of $A \rightarrow B$ and that $\sim A$ is a subformula of $\neg A$, or adjust the calculus slightly (at the expense of some extra search). But it does have the termination property, by an extension of the argument for **GLS'** above. Countermodel construction from failed searches seems to be routine.

But, the first proof found of the type of the S combinator has 5,185 lines!

23 Labelled calculi

Many authors (Kanger, Maslov, Vigano, Castellini, Russo, Fitting, Simpson, Negri, Schmidt, Tishkovsky, ...) have exploited labels (aka "prefixes") in sequent calculi (or tableau calculi), one motivation being to make the inference rules invertible (and another being to allow uniform development of analytic calculi from frame conditions rather than from axioms). Some have criticised this as a lack of syntactic purity; others defend it as allowing calculi for otherwise unmanageable logics. Goré has a useful survey [23] in the context of modal logics.

Using labelled tableaux, Schmidt and Tishkovksy have implemented a generic tableau calculus generator [48], geared rather towards description logics; this can generate a JAVA-based prover, or could be combined with a tableau-based theorem prover such as LOTREC [36] or the Tableaux Work Bench [1].

For **Int**, and using sequent calculus notation rather than tableaux, one statement of the method is by RD and Negri [13]; this covers a wide range of intermediate logics (all those where the first-order frame conditions can be presented as geometric implications, i.e. all [14] that can be presented semantically using first-order formulae).

This approach solves the problem of backtracking; but termination is a problem, with various approaches (including the "unrestricted blocking" rule of [48] and another method in [21, 42]).

24 Calculus G3I

The calculus just mentioned (by RD and Negri [13]) is as follows:

$$\frac{}{x : \bot, \Gamma \Longrightarrow \Delta}\ L\bot \qquad \frac{}{x \leq y, x : P, \Gamma \Longrightarrow \Delta, y : P}\ Ax$$

$$\frac{x : A, x : B, \Gamma \Longrightarrow \Delta}{x : A \wedge B, \Gamma \Longrightarrow \Delta}\ L\wedge \qquad \frac{\Gamma \Longrightarrow \Delta, x : A \quad \Gamma \Longrightarrow \Delta, x : B}{\Gamma \Longrightarrow \Delta, x : A \wedge B}\ R\wedge$$

$$\frac{x : A, \Gamma \Longrightarrow \Delta \quad x : B, \Gamma \Longrightarrow \Delta}{x : A \vee B, \Gamma \Longrightarrow \Delta}\ L\vee \qquad \frac{\Gamma \Longrightarrow \Delta, x : A, x : B}{\Gamma \Longrightarrow \Delta, x : A \vee B}\ R\vee$$

$$\frac{x \leq y, x : A \to B, \Gamma \Longrightarrow \Delta, y : A \quad x \leq y, x : A \to B, y : B, \Gamma \Longrightarrow \Delta,}{x \leq y, x : A \to B, \Gamma \Longrightarrow \Delta}\ L\to \qquad \frac{x \leq y, y : A, \Gamma \Longrightarrow \Delta, y : B}{\Gamma \Longrightarrow \Delta, x : A \to B}\ R\to$$

$$\frac{x \leq x, \Gamma \Longrightarrow \Delta}{\Gamma \Longrightarrow \Delta}\ Ref \qquad \frac{x \leq z, x \leq y, y \leq z, \Gamma \Longrightarrow \Delta}{x \leq y, y \leq z, \Gamma \Longrightarrow \Delta}\ Trans$$

with y fresh in $R\to$, i.e. not occurring in the conclusion. Derivations can be restricted to those in which the label x used in the Ref rule already occurs in the conclusion.

This calculus does not terminate (e.g. on Peirce's formula). Negri [42] shows how to add a loop-checking mechanism to ensure termination. The effect on complexity isn't yet clear; but the loop-checking is expensive.

25 Calculus G3ip^{lab}

We present a calculus closer to an implementation of the above, with sequents having several components: Π for antecedent labelled atoms, Θ for antecedent labelled implications, and so on. The rules being invertible, we can try them in any order; we choose to try the uppermost rules first, and only deal with those (the last four) involving the accessibility relation last. We don't make that relation explicit, but prefer to deal with its consequences asap. A different approach would be needed for all the extensions with geometric rules for intermediate logics.

$$\frac{}{x:P,\Pi;\ \Theta;\ \Gamma \Longrightarrow \Delta;\ \Psi;\ x:P,\Sigma}\ Ax1 \qquad \frac{}{\Pi;\ x:A,\Theta;\ \Gamma \Longrightarrow \Delta;\ x:A,\Psi;\ \Sigma}\ Ax2$$

$$\frac{}{\Pi;\ \Theta;\ x:\bot,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ L\bot \qquad \frac{\Pi;\ \Theta;\ \Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ \Gamma \Longrightarrow x:\bot,\Delta;\ \Psi;\ \Sigma}\ R\bot$$

$$\frac{x:P,\Pi;\ \Theta;\ \Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ x:P,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ LAt \qquad \frac{\Pi;\ \Theta;\ \Gamma \Longrightarrow \Delta;\ \Psi;\ x:P,\Sigma}{\Pi;\ \Theta;\ \Gamma \Longrightarrow x:P,\Delta;\ \Psi;\ \Sigma}\ RAt$$

$$\frac{\Pi;\ \Theta;\ x:A,x:B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ x:A\wedge B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ L\wedge \qquad \frac{\Pi;\ \Theta;\ \Gamma \Longrightarrow x:A,\Delta;\ \Psi;\ \Sigma \quad \Pi;\ \Theta;\ \Gamma \Longrightarrow x:B,\Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ \Gamma \Longrightarrow x:A\wedge B,\Delta;\ \Psi;\ \Sigma}\ R\wedge$$

$$\frac{\Pi;\ \Theta;\ x:A,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma \quad \Pi;\ \Theta;\ x:B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ x:A\vee B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ L\vee \qquad \frac{\Pi;\ \Theta;\ \Gamma \Longrightarrow x:A,x:B,\Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ \Gamma \Longrightarrow x:A\vee B,\Delta;\ \Psi;\ \Sigma}\ R\vee$$

$$\frac{\Pi;\ \Theta;\ \Gamma \Longrightarrow \Delta;\ x:A{\to}B,\Psi;\ \Sigma}{\Pi;\ \Theta;\ \Gamma \Longrightarrow x:A\to B,\Delta;\ \Psi;\ \Sigma}\ R{\to} \qquad \frac{\Pi;\ \Theta;\ \Gamma \Longrightarrow \Delta;\ x:\neg A,\Psi;\ \Sigma}{\Pi;\ \Theta;\ \Gamma \Longrightarrow x:\neg A,\Delta;\ \Psi;\ \Sigma}\ R\neg$$

$$\frac{\Pi;\ x:A{\to}B,\Theta;\ \Gamma \Longrightarrow x:A,\Delta;\ \Psi;\ \Sigma \qquad \Pi;\ x:A{\to}B,\Theta;\ x:B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ x:A{\to}B,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ L{\to}$$

$$\frac{\Pi;\ x:\neg A,\Theta;\ \Gamma \Longrightarrow x:A,\Delta;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ x:\neg A,\Gamma \Longrightarrow \Delta;\ \Psi;\ \Sigma}\ L\neg$$

$$\frac{\Pi;\ \Theta;\ y:A,\Theta^y_x \Longrightarrow y:B;\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ [] \Longrightarrow [];\ x:A\to B,\Psi;\ \Sigma}\ RR{\to} \qquad \frac{\Pi;\ \Theta;\ y:A,\Theta^y_x \Longrightarrow [];\ \Psi;\ \Sigma}{\Pi;\ \Theta;\ [] \Longrightarrow [];\ x:\neg A,\Psi;\ \Sigma}\ RR\neg$$

In the rules $RR{\to}$ and $RR\neg$, it is required that Π and Σ are disjoint and Θ is disjoint from (respectively) $x:A\to B,\Psi$ and from $x:\neg A,\Psi$. Θ^y_x is a copy of all the x-labelled formulae from Θ, relabelled with y (which must be fresh).

Like the previous calculus, this calculus does not terminate.

26 Focused calculi

Naive implementations of the calculi mentioned above spend a great deal of time looking along lists to find a formula of a certain form; a better approach is to take the next formula and either analyse it (i.e. generate appropriate subproblems) or put it aside in a suitable place for later use. For example, atomic formulae can be examined (to see if the branch closes) or (if that fails) put into a list of atoms; and succedent conjunctions put aside until all non-branching rules have been dealt with. This can be regarded as a naive form of *focusing*. So can, to some extent, the calculus **G4ip**, with its connections to **LJQ**.

But several authors, notably McLaughlin and Pfenning [38], have more logic-based approaches. For lack of time, we omit their presentation.

27 Challenges and open problems

1. Find a simple calculus for **Int** that (a) has the termination property (ideally, with linear depth) and (b) avoids backtracking through rules, but without implementing the usual meta-level "list of disjunctive goals to be tried one after another". This can be done for classical logic and for Gödel-Dummett logic [12]. Is there a fundamental complexity result (yet to be discovered) that forbids this?

2. Is there a combination of the **G4ip** ideas and labelling that solves this problem ?

3. Find, develop and simplify uniform methods for ensuring termination in labelled calculi.

4. Find syntactic (i.e. non-semantic) methods for proving cut admissibility for calculi with sequents with several components, e.g. **LSJ** and **GLJ**.

5. Is there a calculus that combines the good features of **G4ip** (where it is the nested implications—formulae of type (4)—that are problematic) and Mint's resolution method (where these are less of a problem: his class [0,1,2,4,5,7] is in P). Or do we get the bad features of both ?

6. Develop more proofs of correctness and completeness using proof assistants like *NuPRL*, *Coq* and *Agda*, extending work of Underwood [53], Caldwell [4], Weich [57, 58] and allowing extraction of verified software in (e.g.) Haskell, Scheme or OCaml. There is some recent work (unpublished) by Larchey-Wendling on **LSJ** (and on **G4ip**) in this direction.

References

[1] P. Abate, R. Goré. *The Tableaux Work Bench*, Proceedings of IJCAR 2003, LNCS **2796**, pp 230–236, 2003.

[2] R. Antonsen, A. Waaler. *A labelled system for IPL with variable splitting*, Proceedings of CADE 2007, LNAI **4603**, pp 132–146, 2007.

[3] E. W. Beth. *The foundations of mathematics*, North-Holland, 1959.

[4] J. Caldwell. *Decidability extracted: synthesizing "correct-by-construction" decision procedures from constructive proofs"*, PhD dissertation, Cornell University, 1998

[5] G. Corsi, G. Tassi. *Intuitionistic logic freed of all metarules*, J. Symb. Logic **72**, pp 1204–1218, 2007.

[6] K. Došen. *A note on Gentzen's decision procedure for intuitionistic propositional logic*, Zeitschrift für mathematische Logik und Grundlagen der Mathematik **33**, pp. 453–456, 1987.

[7] A. G. Dragalin. *Mathematical intuitionism*, Translations of Mathematical Monographs **67**. Translated by E. Mendelson. American Mathematical Society, Providence, R. I., 1988.

[8] R. Dyckhoff. *Contraction-free sequent calculi for intuitionistic logic*, J. Symb. Logic **57**, pp 795–807, 1992.

[9] R. Dyckhoff, D. Kesner, S. Lengrand. *Strong cut-elimination systems for Hudelmaier's depth-bounded sequent calculus for implicational logic*, in IJCAR 2006 Proceedings, LNCS **4130**, pp 347–361, 2006.

[10] R. Dyckhoff, S. Lengrand. *LJQ: a strongly focused calculus for intuitionistic logic*, Computability in Europe 2006 Proceedings, LNCS **3988**, pp 173–185, 2006.

[11] R. Dyckhoff, S. Negri. *Admissibility of structural rules for contraction-free systems of intuitionistic logic*, J. Symbolic Logic **65**, pp 1499–1518, 2000.

[12] R. Dyckhoff, S. Negri. *Decision methods for linearly ordered Heyting algebras*, Arch. Math. Log. **45**, pp 411–422, 2006.

[13] R. Dyckhoff, S. Negri. *Proof analysis for intermediate logics*, Arch. Math. Log. **51**, pp 71–92, 2012.

[14] R. Dyckhoff, S. Negri. *Geometrisation of first-order formulae*, in preparation, 2013.

[15] U. Egly, S. Schmitt. *On intuitionistic proof transformations, their complexity, and application to constructive program synthesis*, Fundamenta Informatica **39**, pp 59–83, 1999.

[16] M. Ferrari, C. Fiorentino, G. Fiorino. *Contraction-free linear depth sequent calculi for intuitionistic propositional logic with the subformula property and minimal depth counter-models*, J. Automated Reasoning **51**, pp 129–149, 2013.

[17] M. Ferrari, C. Fiorentino, G. Fiorino. *Simplification rules for intuitionistic propositional tableaux*, ACM Trans. Comput. Log. **13**, 14 pp, 2012.

[18] M. Fitting. *Intuitionistic logic, model theory and forcing*, North-Holland, 1969.

[19] M. Fitting. *Proof methods for modal and intuitionistic logic*, North-Holland, 1969.

[20] T. Franzén. *Algorithmic aspects of intuitionistic propositional logic, I and II*, SICS Research Reports R870X and R8906, 1987 and 1989.

[21] D. Garg, V. Genovese, S. Negri. *Countermodels from sequent calculi in multi-modal logics* LICS 2012 Proceedings, IEEE, pp 315–324, 2012.

[22] G. Gentzen. *Untersuchungen über das logische Schliessen*, Math. Zeitschrift **39**, pp 176–210, 405–431, 1935.

[23] R. Goré. *Tableau methods for modal and temporal logics*, in Handbook of Tableau Methods, Kluwer, pp 297–396, 1999.

[24] R. Goré, J. Thomson. *BDD-based automated reasoning in propositional non-classical logics: progress report*, PAAR 2012, CEUR Workshop Proceedings, to appear, 15 pp, 2012.

[25] J. Goubault-Larrecq. *Implementing tableaux by decision diagrams*, Unpublished note, 47pp, 1996.

[26] H. Herbelin. *Séquents qu'on calcule*, Thèse de Doctorat, Université Paris 7, 1995.

[27] A. Heuerding, M. Seyfried, H. Zimmermann. *Efficient loop-check for backward proof search in some non-classical logics*, in Tableaux 1996 Proceedings, LNAI **1071**, pp 210–225, 1996.

[28] J. Howe. *Two loop-detection mechanisms: a comparison*, in Galmiche (ed), Tableaux 1997 Proceedings, LNCS **1227**, pp 188–200, 1997.

[29] J. Hudelmaier. *A Prolog program for intuitionistic propositional logic*, SNS-Bericht **88-28**, Tübingen, 1988.

[30] J. Hudelmaier. *Bounds for cut elimination in intuitionistic propositional logic*, Arch. Math. Logic **31**, pp 331–353, 1992.

[31] J. Hudelmaier. *An O(n log n)-space decision procedure for intuitionistic propositional logic*, J. Logic and Computation **3**, pp 63–76, 1993.

[32] O. Ketonen. *Untersuchungen zum Prädikatenkalkül*, Annales Acad. Sci. Fenn, Ser. A.I. **23**, 1944.

[33] S.C. Kleene. *Introduction to metamathematics*, North-Holland, 1952.

[34] D. Larchey-Wendling, D. Mery, D. Galmiche. *STRIP: Structural sharing for efficient proof-search*, IJCAR 2001, LNCS **2083**, pp 696-700, 2001.

[35] P. Lincoln, A. Scedrov, N. Shankar. *Linearizing intuitionistic implication*, Annals of Pure and Applied Logic **60**, pp 151–177, 1993.

[36] O. Gasquet, A. Herzig, D. Longin, and M. Sahade. *LoTREC: Logical tableaux research engineering companion*, Proceedings of TABLEAUX05, LNCS **3702**, pp 318–322, 2005.

[37] S. Maehara. *Eine Darstellung der Intuitionistischen Logik in der Klassischen*, Nagoya Math. J. **7**, pp 45–64, 1954.

[38] S. McLaughlin, F. Pfenning. *Imogen: Focusing the polarized inverse method for intuitionistic propositional logic*, in Proceedings of LPAR'08, LNCS **5330**, pp 174–181, 2008.

[39] G. Mints. *Gentzen-type systems and resolution rule, Part I.* LNCS **417**, pp 198–231, 1990.

[40] G. Mints. *Complexity of subclasses of the intuitionistic propositional calculus*, Programming Logic (ed B. Nordström), BIT **31**, pp 64–69, 1992.

[41] G. Mints. *A short introduction to intuitionistic logic*, CSLI Stanford Lecture Notes, Springer, 2000.

[42] S. Negri. *On the duality of proofs and countermodels in labelled sequent calculi*, in TABLEAUX' 2013, LNCS **8123**, pp 5–9, 2013

[43] J. Otten. *Clausal Connection-Based Theorem Proving in Intuitionistic First-Order Logic*, TABLEAUX 2005, LNCS **3702**, pp 245–261, 2005.

[44] J. Otten. The ILTP Library, `http://www.cs.uni-potsdam.de/ti/iltp/`

[45] L. Pinto, R. Dyckhoff. *Loop-free construction of counter-models for intuitionistic propositional logic*, in M. Behara, R. Fritsch, R. G. Lintz (eds), "Symposia Gaussiana, Conf. A", Walter de Gruyter & Co, Berlin, pp 225–232, 1995.

[46] D. Sahlin, T. Franzén, S. Haridi *An intuitionistic predicate logic theorem prover*, J. Logic and Computation **2**, pp 619–656, 1992.

[47] G. Sambin, S. Valentini. *The modal logic of provability. The sequential approach*, J. Philos. Logic **11**, pp 311–342, 1982.

[48] R. Schmidt, D. Tishkovsky. *Automated synthesis of tableau calculi*, Logical Methods in Computer Science **7**, 32 pp, 2011.

[49] K. Schütte. *Vollstandige Systeme modaler und intuitionistischer Logik*, Ergebnisse der Mathematik, Springer, 1968.

[50] R. Statman. *Intuitionistic propositional logic is polynomial-space complete*, Theoretical Computer Science **9**, pp 67–72, 1979.

[51] T. Tammet. *A resolution theorem prover for intuitionistic logic*, CADE-13, LNCS **1104**, pp 2–16, Springer, 1996.

[52] A. S. Troelstra, H. Schwichtenberg. *Basic proof theory*, Cambridge, 2001.

[53] J. Underwood. *A constructive completeness proof for intuitionistic propositional calculus*, TR 90-1179, Dept. of Computer Science, Cornell University, 1990; also in Proceedings of the Workshop on Analytic Tableaux, Marseille 1993.

[54] N. N. Vorob'ev. *The derivability problem in the constructive propositional calculus with strong negation*, Doklady Akademii Nauk SSSR **85**, pp 689–692, 1952.

[55] N. N. Vorob'ev. *A new algorithm for derivability in the constructive propositional calculus*, AMS Translations Ser. 2 **94**, pp 37–71, 1970.

[56] A. Waaler, L. Wallen. *Tableaux methods in intuitionistic logic*, in M. D'Agostino, D. M. Gabbay, R. Hähnle, J. Posegga (eds). "Handbook of Tableaux Methods", Kluwer, Dordrecht, March 1999, pp 255–296, 1999.

[57] K. Weich. *Improving proof search in intuitionistic propositional logic*, Munich PhD thesis, also from Logos Verlag Berlin, 2001.

[58] K. Weich. *Decision procedures for intuitionistic propositional logic by program extraction*, Tableaux 1998 Proceedings, LNCS **1397**, pp 292–306, 1998.