Two Examples of Cut-Elimination for Non-classical Logics

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System S4C

Dynamic Topological Logic: $\Box, \circ$.
$S4$-axioms for the $\Box$; $\circ$ commutes with Boolean connectives:

$$\circ(A \to B) \leftrightarrow (\circ A \to \circ B), \qquad \circ\bot \leftrightarrow \bot$$

$\circ\Box A \to \Box \circ A$ (the axiom of continuity)

Inference Rules

Standard axioms: $A, \Gamma \Rightarrow \Delta, A, \qquad \bot, \Gamma \to \Delta.$
Box-formulas are ones of the form $\circ^k \Box A$, $k \geq 0$.
Boldface letters $\mathbf{B}, \mathbf{C}, \mathbf{D}$ stand for *Box*-formulas.

$$\frac{\circ^k A, \Gamma \Rightarrow \Delta, \circ^k B}{\Gamma \Rightarrow \Delta, \circ^k (A \to B)} \Rightarrow\to \qquad \frac{\Gamma \Rightarrow \Delta, \circ^k A \quad \circ^k B, \Gamma \Rightarrow \Delta}{\circ^k (A \to B), \Gamma \Rightarrow \Delta} \to\Rightarrow$$

$$\frac{\circ^k A, \Gamma \Rightarrow \Delta}{\circ^k \Box A, \Gamma \Rightarrow \Delta} \Box\Rightarrow \qquad \frac{\Gamma \Rightarrow \Delta}{\circ\Gamma \Rightarrow \circ\Delta} \circ \qquad \frac{\mathbf{B} \Rightarrow A}{\mathbf{B} \Rightarrow \Box A} \Rightarrow\Box$$

Substitution of derivations

The substitution to the right. Given two deductions
from some sequents, some of them non-axioms

$$\Gamma \Rightarrow A, \Delta \qquad \text{and} \qquad A, \Gamma' \Rightarrow \Delta'$$

construct a derivation of $\quad \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$
by climbing the derivation of $A, \Gamma' \Rightarrow \Delta'$,
in general, from sequents obtained by replacing
some antecedent A by $\Gamma \Rightarrow \Delta$.
[Prawitz's distinction]
In non-classical (intuitionistic, modal) systems may be impossible.

The substitution to the left. Given two deductions

$$\Gamma \Rightarrow A, \Delta \qquad \text{and } A, \Gamma' \Rightarrow \Delta'$$

construct a derivation of $\qquad \Gamma, \Gamma' \Rightarrow \Delta, \Delta'$
by climbing the derivation of $\Gamma \Rightarrow \Delta, A$,
in general, from sequents obtained by replacing
some succedent A by $\Gamma \Rightarrow \Delta$.

The problematic instance:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \circ^{k-1}\Box A}{\circ\Gamma \Rightarrow \circ\Delta, \circ^{k}\Box A} \quad \dfrac{\circ^{k}\Box A, \mathbf{B} \Rightarrow C}{\circ^{k}\Box A, \mathbf{B} \Rightarrow \Box C}}{\circ\Gamma, \mathbf{B} \Rightarrow \circ\Delta, \Box C}$$

Lemma (Substitution Lemma)

1. Substitution is admissible for S4C in the form:

$$\dfrac{\mathbf{B} \Rightarrow \Box C \quad \Box C, \Gamma \Rightarrow \Delta}{\mathbf{B}, \Gamma \Rightarrow \Delta}$$

2. Substitution to the left is admissible if the left branch does not contain the $(\circ)$ *rule.*

Direct Derivation

A derivation of a sequent $\Gamma \Rightarrow \Delta, \circ^k \Box A$, $k \geq 0$, is *direct* w.r.t. $\circ^k A$ if it has the form

$$\begin{array}{ccc} F & & \dfrac{\mathbf{D} \Rightarrow A}{\circ^k \mathbf{D} \Rightarrow \circ^k \Box A} \\ \nwarrow & \vdots \text{endpiece} & \nearrow \\ & \Gamma \to \Delta, \circ^k \Box A & \end{array}$$

where the endpiece consists of $\to, \Box \Rightarrow$-inferences, and the sequents denoted by F do not contain formulas traceable to the succedent formula $\circ^k \Box A$.

Lemma

Every cut-free derivation of $\Gamma \Rightarrow \circ^k \Box A$ *can be transformed into a cut-free direct derivation of the same sequent w.r.t.* $\circ^k \Box A$.

Proof. Induction on the derivation. Since the endpiece consists of inferences invariant under adding $\circ$, all $(\circ)$-inferences can be moved up to the boundary of the endpiece. $\dashv$

Theorem (Cut-elimination)

Any derivation in S4C plus cut can be transformed into a derivation without cut.

Proof. As always it is enough to eliminate the uppermost cut of the maximal complexity. In the problematic case when the cut formula is of the form $\circ^k \Box C$ for $k > 0$ the substitution to the left and Lemma above reduce this cut to the form treated in the Substitution Lemma. Resulting recedeces are replaced by cuts of smaller complexity.

The Logic of Transitive and Dense Frames

$$K + \ \Box A \leftrightarrow \Box\Box A$$

The system GTD

$$\frac{\Gamma, \Box\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A} \Rightarrow \Box \qquad \frac{\Gamma, \Box\Gamma \Rightarrow \Box A}{\Box\Gamma \Rightarrow \Box A} \Box \Rightarrow$$

Derivability of GTD rules in $K + \Box A \leftrightarrow \Box\Box A$

$$\dfrac{\dfrac{\dfrac{\Gamma, \Box\Gamma \Rightarrow A}{\Box\Gamma, \Box\Box\Gamma \Rightarrow \Box A}}{\Box\Gamma, \Box\Gamma \Rightarrow \Box A}}{\Box\Gamma \Rightarrow \Box A} \qquad \dfrac{\dfrac{\dfrac{\Gamma, \Box\Gamma \Rightarrow \Box A}{\Box\Gamma, \Box\Box\Gamma \Rightarrow \Box\Box A}}{\Box\Gamma, \Box\Gamma \Rightarrow \Box A}}{\Box\Gamma \Rightarrow \Box A}$$

The cut rule is replaced by

$$\frac{\Gamma \Rightarrow \Delta \quad \Sigma \Rightarrow \Pi}{\Gamma, \Sigma_C \Rightarrow \Delta_C, \Pi}\ mix$$

which is permuted up the derivation till each of the mix-formulas is inferred by the very last rule. Then the complexity (of the mix-formula C) is reduced.

Cuts over atomic formulas and Boolean connectives are unproblematic.

Now consider mix-formulas beginning with a $\Box$. Note that a conclusion of any $\Box$-rule has the form $\Box\Gamma \Rightarrow \Box\delta$.

Lemma

Consider a mix inference

$$\frac{\Gamma \Rightarrow \Delta \quad \Sigma \Rightarrow \Pi}{\Gamma, \Sigma_{\Box B} \Rightarrow \Delta_{\Box B}, \Pi}$$

Suppose $\mathcal{L}$ is an adjacent inference which does not change mix-formulas $\Box B$. Then the mix inference can be permuted over $\mathcal{L}$ in the following two cases.
Case 1. $\mathcal{L}$ is a structural or Boolean rule.
Case 2. $\mathcal{L}$ is $\Rightarrow \Box$ introducing the l.h.s. premise $\Gamma \Rightarrow \Delta$ and the r.h.s. premise $\Sigma \Rightarrow \Pi$ is introduced by a $\Box$-rule.

Proof. Permutations with structural and Boolean rules are standard.
The only modal rule which does not change $\Box B$ is $\Box \Rightarrow$ when this rule is in the l.h.s. premise.

$$\dfrac{\dfrac{\Gamma, \Box\Gamma \Rightarrow \Box B}{\Box\Gamma \Rightarrow \Box B}\ \mathcal{L} \qquad \Box B, \Box\Sigma \Rightarrow \Box\delta}{\Box\Gamma, \Box\Sigma_{\Box B} \Rightarrow \Box\delta}\ mix$$

New derivation:

$$\dfrac{\dfrac{\dfrac{\Gamma, \Box\Gamma \Rightarrow \Box B \qquad \Box B, \Box\Sigma \Rightarrow \Box\delta}{\Gamma, \Box\Gamma, \Box\Sigma_{\Box B} \Rightarrow \Box\delta}\ mix}{\Gamma, \Sigma_{\Box B}, \Box\Gamma, \Box\Sigma_{\Box B} \Rightarrow \Box\delta}}{\Box\Gamma, \Box\Sigma_{\Box B} \Rightarrow \Box\delta}\ \mathcal{L}$$

$\dashv$

The standard reduction decreasing complexity of the mix-formula works for our system. Given derivation:

$$\dfrac{\dfrac{\Gamma, \Box\Gamma \Rightarrow B}{\Box\Gamma \Rightarrow \Box B} \quad \dfrac{B, \Sigma, \Box B, \Box\Sigma \Rightarrow \delta'}{\Box B, \Box\Sigma \Rightarrow \Box\delta}}{\Box\Gamma, \Box\Sigma_{\Box B} \Rightarrow \Box\delta}$$

where δ' is either δ or $\Box\delta$, is reduced to

$$\dfrac{\dfrac{\dfrac{\Gamma, \Box\Gamma \Rightarrow B \qquad \dfrac{\dfrac{\Gamma, \Box\Gamma \Rightarrow \Box B}{\Box\Gamma \Rightarrow \Box B} \quad B, \Sigma, \Box B, \Box\Sigma \Rightarrow \delta'}{B, \Sigma_{\Box B}, \Box\Gamma, \Box\Sigma_{\Box B} \Rightarrow \delta'}}{\Gamma, \Sigma_{\Box B, B}, \Box\Gamma, \Box\Gamma_{B}, \Box\Sigma_{\Box B, B} \Rightarrow \delta'}}{\Gamma, \Sigma_{\Box B}, \Box\Gamma, \Box\Sigma_{\Box B} \Rightarrow \delta'}}{\Box\Gamma, \Box\Sigma_{\Box B} \Rightarrow \Box\delta}$$

Now mix-elimination theorem is proved by a standard schema.