When Kripke-Platek Set Theory Meets One Powerset

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Dummett argues that classical quantification is illegitimate when the domain is given as the objects which fall under an indefinitely extensible concept.

The Continuum Hypothesis

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Is the continuum hypothesis a definite mathematical problem?

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- **Classical logic** for bounded (Δ_0) formulas. **Intuitionistic logic** for unbounded quantification.

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- $(*)$ follows from the observation that $\mathbf{AC}_{full}$ implies LEM_{Δ_0} (Diaconescu) and also BOS.
- Note that **T** proves full **Replacement** and **Strong Collection** (considered by Tharp, Beeson, Aczel).
- **T** is quite strong. It proves every theorem of (classical) second order arithmetic. In strength it resides strictly between **second order arithmetic** and **Zermelo set theory**.

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- If $\mathbb{R} \notin L$ then $L \neq L(\mathbb{R})$. However, always $L[\mathbb{R}] = L$.
- Only $L[A]$ is interesting for our purposes.

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- $B = A \cap L[A] \Rightarrow L[A] = L[B] \wedge (V = L[B])^{L[A]}$.

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- There is a Σ_1 formula $\mathrm{wo}(x, y, z)$ such that

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  and if $<_{L[A]}$ denotes the wellordering of $L[A]$ determined by wo, then for any limit $\lambda > \omega$,

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- Start with a universe V_0 such that

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  Can be obtained from any universe V' such that $V' \models \mathbf{ZFC} + \mathrm{GCH}$ (e.g. L) by forcing with $\mathrm{Fn}(\kappa \times \omega, 2)$ where $\kappa = (\aleph_2)^{V'}$.

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- We now code the set of reals $\mathbb{R}$ via a set A of ordinals in such a way that the set of real numbers of V_0 belong to $L[A]$. We thus have
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- Clearly,

$$L[A] \models \neg CH.$$

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\begin{array}{rcl}
e \Vdash a \in b & \text{iff} & a \in b \\
e \Vdash a = b & \text{iff} & a = b \\
e \Vdash \varphi \wedge \psi & \text{iff} & (e)_0 \Vdash \varphi \text{ and } (e)_1 \Vdash \psi \\
e \Vdash \varphi \vee \psi & \text{iff} & [(e)_0 = 0 \wedge (e)_1 \Vdash \varphi] \text{ or } [(e)_0 = 1 \wedge (e)_1 \Vdash \psi] \\
e \Vdash \varphi \rightarrow \psi & \text{iff} & \forall d\, [d \Vdash \varphi \Rightarrow e \bullet d \Vdash \psi] \\
e \Vdash \exists x \theta(x) & \text{iff} & (e)_1 \Vdash \theta((e)_0) \\
e \Vdash \forall x \theta(x) & \text{iff} & \forall a \in L[A]\; e \bullet a \Vdash \theta(a).
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Theorem 1. The structures we do realizability over are pca's or models of **App.**
Whenever

$$\mathbf{T} \vdash \psi$$

then we can effectively construct an applicative term t from the derivation such that for all structures $\langle L[A], \mathbb{R}^{V_0}, \in, A \rangle$ viewed as models of **App**, we have $t\downarrow$ in $L[A]$, i.e.

$$L[A] \models \exists e\,[t \simeq e \wedge e \Vdash \theta].$$

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- Since $L[A] \models \neg CH$ we must have

$$L_\pi[A] \models (e)_0 = 1 \wedge L[A] \models \forall d\, d \not\Vdash CH.$$

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- $L[A \cup B] \models CH$.

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- V_1 has a bijection h between $\mathbb{R}$ and $\aleph_1$. Code h into a set of ordinals B such that $B \cap \pi = \emptyset$.
- $L[A \cup B] \models CH$.
- $L[A \cup B] \models \exists d\, d \Vdash CH$.
- $L[A \cup B] \models (e)_0 = 0$.

Proving the conjecture cont'ed

- Take a forcing extensions V_1 of V_0 such that
$$V_1 \models 2^{\aleph_0} = \aleph_1$$
and
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Proving the conjecture cont'ed

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- **CONTRADICTION!**

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Goethe started writing **Dichtung und Wahrheit** in 1809 when he was sixty.

Thank you!