A Miniaturized "Predicativity" – "slow growing" analogues.

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§0. Input–Output Theories for Grzegorczyk Hierarchy.

- $EA(I; O)$ is a 2-sorted theory with elementary strength.
- $EA(I; O) \subset EA(I; O)^+ \vdash \mathcal{E}^3(x : I) \downarrow .$
- $EA(I_1; O)^+(I_2)^+ \vdash \mathcal{E}^4(x : I_2) \downarrow .$
- $EA(I_1, I_2, \ldots, I_k; O)^+ \vdash \mathcal{E}^{k+2}(x : I_k) \downarrow .$
- $EA(I_1, I_2, \ldots I_\omega; O)^\infty_{\prec\Gamma_0} \vdash \mathcal{E}^\omega(x : I_\omega) \downarrow .$

The Main Principles:
(1) Numerical inputs govern induction-length.
(2) Values computable from inputs only may be used as input.
(3) There may be many "increasingly refined" levels of input.

§1. $EA(I; O)$ – Leivant (1995), Ostrin-Wainer (2005),

- Quantified numerical "output" variables $a, b, c, \ldots$.
- Unquantified "input" variables $x, y, z, \ldots$ (constants).
- Terms $0, Succ, +, \times, \pi, \pi_0, \pi_1, \ldots$ with usual axioms.
- "Predicative/bounded/pointwise Induction" up to x:

  $$A(0) \wedge \forall a(A(a) \to A(a+1)) \to \forall a \leq x A(a).$$

- Define $f(x)\downarrow \equiv \exists a C_f(x, a)$ for some Σ_1 formula C_f.
- Then $EA(I; O) \vdash f(x)\downarrow$ if and only if f is elementary.

$EA(I;O) \subset EA(I;O)^+$ – Spoors-Wainer (2012)

$EA(I;O)$ is not "user-friendly" since composition of functions $f : I \rightarrow O$ cannot be proved straightforwardly – however Wirz (2005) developed a variety of derived rules showing this.
To remedy this, add a Σ_1-"Reflection Rule" as in Cantini (2002):

$$\frac{\Sigma(\vec{x}), \exists a A(a, \vec{x})}{\Sigma(\vec{x}), \exists y A(y, \vec{x})}$$

where the only free parameters are inputs $\vec{x}$. And add I-quantifiers:

$$\frac{\Gamma, A(x)}{\Gamma, \forall y A(y)} \qquad \frac{\Gamma, A(t(\vec{x}))}{\Gamma, \exists y A(y)}.$$

Note: the inductions are still restricted to $EA(I;O)$ formulas only.

Then if $\vdash f(x)\downarrow$ and $\vdash g(x)\downarrow$ we can directly prove $\forall y f(y)\downarrow$ and (by reflection) $\exists y(g(x) = y)$. Therefore $EA(I;O)^+ \vdash f(g(x))\downarrow$.

§2. $EA(I_1, I_2; O) = EA(I_1; O)^+(I_2)^+$.

Add to $EA(I_1; O)^+$ a new layer of I_2–inputs $u, v, \ldots$ and a new level of inductions:

$$A(0) \wedge \forall a(A(a) \rightarrow A(a+1)) \rightarrow A(u)$$

where A is now any $EA(I_1; O)^+$ formula.

Then:

- $EA(I_1; O) \vdash 2^x\downarrow$
- $EA(I_1; O)^+ \vdash \forall x \exists y(2^x = y)$
- $EA(I_1; O)^+ \vdash \exists y(2^x_a = y) \rightarrow \exists y(2^x_{a+1} = y)$
- $EA(I_1; O)^+(I_2) \vdash \forall x \exists y(2^x_u = y)$

Now add I_2–quantifier rules and a Σ_1–reflection rule for I_2. This allows compositions of the superexponential etc., so

$$EA(I_1; O)^+(I_2)^+ \vdash \mathcal{E}^4(u)\downarrow \ .$$

§3. Pointwise Transfinite Induction.

Usual transfinite induction TI(A, α) may be written:

$$A(0) \wedge \forall\gamma(A(\gamma) \rightarrow A(\gamma+1)) \wedge \forall\lambda(\forall i A(\lambda_i) \rightarrow A(\lambda)) \rightarrow A(\alpha)$$

Definition

"Weak, pointwise transfinite induction" PTI(A_x,α):

$$A(0) \wedge \forall\gamma(A(\gamma) \rightarrow A(\gamma+1)) \wedge \forall\lambda(\forall i \leq x A(\lambda_i) \rightarrow A(\lambda)) \rightarrow A(\alpha)$$

or $\quad A(0) \wedge \forall\gamma(A(\gamma) \rightarrow A(\gamma+1)) \wedge \forall\lambda(A(\lambda_x) \rightarrow A(\lambda)) \rightarrow A(\alpha).$

The idea goes back to U. Schmerl (1982).

This is enough to define the Slow Growing function $G_\alpha(x)$ and is equivalent to Bounded Induction "up to" $G_\alpha(x)$:

$$A(0) \wedge \forall a(A(a) \rightarrow A(a+1)) \rightarrow \forall a \leq G_\alpha(x) A(a).$$

Tree Ordinals $\alpha \prec \Gamma_0$ and their G_α's.

Definition

- $\alpha \in \Omega$ if $\alpha = 0$ or $\exists \beta \in \Omega(\alpha = \beta + 1)$ or $\alpha : N \to \Omega$.
- $G : N \times \Omega \to N$ is given by

$$G_n(0) = 0, \quad G_n(\beta + 1) = G_n(\beta) + 1. \quad G_n(\lambda) = G_n(\lambda_n) .$$

- $\psi : \Omega \times \Omega \to \Omega$ is given by

$$\psi_0(\beta) = \beta + 2^\beta, \quad \psi_{\alpha+1}(\beta) = \psi_\alpha^{2^\beta}(\beta), \quad \psi_\lambda(\beta) = \sup \psi_{\lambda_n}(\beta) .$$

- $F_0(m) = m + 2^m,\ F_{n+1}(m) = F_n^{2^m}(m)$. Then $F_\omega(n) = F_n(n)$.

Theorem

(i) $|\psi_\alpha(\omega)| =$ *Veblen* $\phi_\alpha(0)$ *for* $\alpha \succ 0$.

(ii) $G_n(\psi_\alpha(\beta)) = F_{G_n(\alpha)}(G_n(\beta))$. *So* $G_n(\psi_\alpha(\alpha)) = F_\omega(G_n(\alpha))$.

$PTI(\alpha)$ in $EA(I, ..; O)$ theories.

- Recall: For $\alpha \prec \varepsilon_0$, $G(\alpha) \in \mathcal{E}^3$, but $G(\varepsilon_0) \notin \mathcal{E}^3$.
- Hence: $EA(I; O) \vdash PTI(\alpha \prec \varepsilon_0)$, but $EA(I; O) \nvdash PTI(\varepsilon_0)$.
- Thus: $\|EA(I; O)^+\|_W = \varepsilon_0 = \phi_1(0)$.
- Similarly: $\|EA(I_1, I_2; O)^+\|_W = \phi_2(0)$ etcetera.
- And: $\|EA(I_1, \ldots, I_\omega; O)^\infty_{\prec\Gamma_0}\|_W = \Gamma_0$.
- Wainer-Williams (2005): $\|ID_1(I; O)\|_W = \phi_{\varepsilon_{\Omega+1}}(0)$ but $ID_1(I; O) \equiv PA$.

Note: Jäger-Probst (2013) and Ranzi-Strahm (2013), SID_ν have full (unstratified) numerical induction in the base theory.

§4. $EA(I_1, I_2, \ldots, I_\omega; O)^{\infty}_{\prec\Gamma_0}$.

Sequents are: $n : I_k \; ; \; \ldots, \; m : I_i \; \vdash^{\alpha} \; \Gamma$ with $\omega \geq k > \ldots > i$.

Logic Rules are as follows $\boxed{\text{where } \beta \prec_n \alpha \prec \Gamma_0}$:

$$(\exists I_i)\frac{n : I_k \; ; \; \ldots, m : I_i \; \vdash^{\beta}_{C} \ell \qquad n : I_k \; ; \ldots, m : I_i \; \vdash^{\beta} \Gamma, A(\ell)}{n : I_k \; ; \ldots, m : I_i \; \vdash^{\alpha} \Gamma, \exists x(I_i(x) \wedge A(x))}$$

$$(\forall I_i)\frac{\{ \; n : I_k \; ; \; \ldots, \max(m, j) : I_i \; \vdash^{\beta} \Gamma, A(j) \; \}_j}{n : I_k \; ; \; \ldots, m : I_i \; \vdash^{\alpha} \Gamma, \forall x(I_i(x) \rightarrow A(x))} \; \text{level}(A) < k$$

and $(\vee), (\wedge)$ and (Cut) as usual, together with Computation Rules:

(Ax) $n; \ldots, m \vdash^{\alpha}_{C} \ell$ if $\ell \leq q(m)$ $\quad (C)\dfrac{n; \ldots m \vdash^{\beta}_{C} m' \qquad n; \ldots m' \vdash^{\beta}_{C} \ell}{n; \ldots m \vdash^{\alpha}_{C} \ell}$

Reading Off Bounding Functions.

Ordinal assignment is "slow growing": $|\{\beta : \beta \prec_n \alpha\}| = G_n(\alpha)$.

Lemma

If $n; m \vdash^{\alpha}_{C} k$ *then* $k \leq q^{G_n(2^\alpha)}(m)$.

Theorem (Basic bounding principle)

If $EA(I;O)^{+} \vdash f(x){\downarrow}$ *then, by embedding and cut-reduction, there is an* $\alpha \prec \varepsilon_0$ *such that for every* $x := n$, $n; - \vdash^{\alpha}_{0} \exists a C_f(n,a)$. *Then* $\exists a \leq k C_f(n,a)$ *where* $k = q^{G_n(2^\alpha)}(0)$. *So* $f \in \mathcal{E}^3$.

Lemma ($\mathcal{E}^4$ bounding)

Let $B_1(\alpha, n) = q^{G_n(2^\alpha)}(0)$ *be the bounding function at level* 1.
Then $B_2(\alpha, n) = B_1(\alpha)^{G_n(2^\alpha)}(n)$ *is the bound at level* 2*:*

$$n_2; n_1, - \vdash^{\alpha}_{C} k \quad \Rightarrow \quad k \leq B_2(\alpha, \max(n_2, n_1))\,.$$

This bound is $\mathcal{E}^4$*-definable. Etcetera.*

§5. Level ω – Ackermann.

Suppressing ordinal bounds, $EA(I_1, I_2, \ldots, I_\omega; O)^+_\infty$ proves:

$$\forall x^r \exists y^r (F_r^{2^a}(x) = y) \to \forall x^r \exists y^r (F_r^{2^{a+1}}(x) = y)$$

Hence by induction on a, using repeated cuts:

$$k : I_{r+1}; \vdash \forall x^r \exists y^r (F_r(x) = y) \to \forall x^r \exists y^r (F_r^{2^k}(x) = y)$$

By Cut on $\forall x^r \exists y^r (F_r(x) = y)$ using $k : I_{r+1} \vdash_C k : I_r$,

$$k : I_{r+1} \vdash \exists y^r (F_{r+1}(k) = y)$$

Then $k : I_{r+1} \vdash \exists y^{r+1} (F_{r+1}(k) = y)$ so $\forall x^{r+1} \exists y^{r+1} (F_{r+1}(x) = y)$.

So $\quad r : I_\omega \vdash \forall x^r \exists y^r (F_r(x) = y)$.

But note: $r : I_\omega \vdash_C r : I_r$. Therefore $\quad r : I_\omega \vdash \exists y^\omega (F_\omega(r) = y)$.

§6. "Predicativity" in $EA(I_1, \ldots, I_\omega; O)^{\infty}_{\prec\Gamma_0}$.

- Sequents are: $n : I_k; ..m : I_i \vdash^{\alpha} \Gamma$ where ordinal bounds $\alpha \prec \Gamma_0$ are autonomously generated according to the rule: $PTI(\beta) \Rightarrow PTI(\psi_\beta(\beta))$.

- This holds because if $G_n(\beta)$ is computable in the system, so is

$$G_n(\psi_\beta(\beta)) \; = \; F_{G_n(\beta)}(G_n(\beta)) \; = \; F_\omega(G_n(\beta)) \; .$$

  Only finite iterations of F_ω are possible, so can't reach Γ_0.

- Collapsing Principle: $n : I_\omega; m : I_i \vdash^{\alpha}_{C} k \quad \Rightarrow \quad m : I_i; \vdash^{a}_{C} k$ where $a = G_n(\psi_\alpha(\alpha)) = F_\omega(G_n(\alpha))$.
  Computational bounds are finite F-terms, elementary in F_ω.

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$$EA(I_1, \ldots I_\omega; O)^{\infty}_{\prec\Gamma_0} \; \vdash \; \mathcal{E}^{\omega}(x : I_\omega) \downarrow$$

$$\| EA(I_1, \ldots I_\omega; O)^{\infty}_{\prec\Gamma_0} \|_W = \Gamma_0 \; .$$

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